Romanian Journal of Regional Science
Vol. 19, No. 2, Winter 2025

DOI: https://doi.org/10.61225/rjrs.2025.09

A MATHEMATICAL MODEL FOR POPULATION DISTRIBUTION III: AN ANALYTICAL APPROACH TO PREY-PREDATOR SYSTEMS

Nicholas Elias

Democritus University of Thrace, GREECE

Corresponding author: Nicholas Elias

Address: 16 Aristophanous str. Kastella, 18533 Piraeus, Attica GREECE

E-mail: nelias@teemail.gr

Biographical Note:

Nicholas Elias received his Ph.D. in Civil Engineering and Applied Mathematics from the School of Engineering of Democritus University where he was accepted as a non-tenured Lecturer. He is also a freelance Engineer (studies and constructions) and technical expert consultant in the Court of Justice of Piraeus. ORCID ID: https://orcid.org/0009-0002-8358-6934

Abstract

The present paper is a continuation of the work described in Elias (2023) and Elias (2024) which attempt to develop a deterministic simulation for phenomena occurring in Regional Science, incorporating demographic, economic, geographical etc. variables to a mathematically simulated deterministic system. Herein, such systems are approximated by a generalization of the Helmholtz wave equation, providing the possibility to better understand the dynamic influence of the environment and of the interaction between the variables within the system, thus making possible the discrimination between inertial, isolated (closed) and (open-general) dynamic cases. A simulation concerning population systems is presented, applied on prey – predator systems, without utilizing the Lotka – Volterra equations. For these systems the analytically derived expressions for the equations of motion (temporal) and distribution (spatiotemporal) are produced.

Keywords: Pray – predator, population distribution, Helmholtz wave equation.
JEL Classification: Y80

1. Introduction – summary of the model

This section summarizes the main results of the model presented in Elias (2023), which intends to develop a deterministic approach for population systems, meaning that the optimum final result of the model should be formatted as a set or differential equations connecting the population variables to the time (equation of motion) or, in general, to the time and the geographical space (equation of distribution). Since the behavior of any population is affected by economic, cultural or geographical variables as well as demographic ones, the above model should be expandable and versatile enough to incorporate a large set of diverse social variables and, at the same time, to involve the least possible number of arbitrary principles.

The first axiomatic principle establishes a proper space of reference by which the system can be adequately described. To achieve this, it is assumed that a subset $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$ of the set of all the variables of the system can always be found such that $\boldsymbol{Q}$ consists of linearly independent members and it adequately describes the behavior of the system. Thus, $\boldsymbol{Q}$ is a base of the space of reference, Q^i are the components of this base and N is the dimension of the space. It should be noticed that the temporal coordinate $x^0 \equiv t$ or the spatiotemporal coordinates $\boldsymbol{x} = \{x^0, x^1, x^2, x^3\}$ which is the usual base of a geographical four-dimensional Euclidean space, are not included in any base $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$ of the space of reference but, rather, as parameters of the derived constitutional equations. Indeed, the behavior of a social system is independent of the values of both time or geographical space, since in a completely empty, homogeneous and isotropic field (territory, habitat), the placement of the same society at different geographical coordinates and time would not produce any variation in the system behavior. A treatise on the influence of topography on the system can be found in Elias (2024).

For the model to produce, as a final result, a set of simultaneous differential equations of the components of the base, the space of reference should be constructed as an Riemannian (metric) N-dimensional space having a metric form: $(ds)^2 = g_{ij} dQ^i dQ^j$ $(i, j = 1, \dots, N)$, where ds is the elementary arc length of any curve imbedded into the space and the functions $g_{ij} = g_{ij}(Q^1, \dots, Q^N)$ are the components of the metric tensor. Hence, each event of the system can be represented by a point of the space of reference, its history coincides with a trajectory curve, imbedded into the space, and the state of the system, that is its Lagrangean, is a function of the form: $L = L(\boldsymbol{Q}(t), d\boldsymbol{Q}(t)/dt) = L(Q^1(t), \dots, Q^N(t), dQ^1(t)/dt, \dots, dQ^N(t)/dt)$.

The deterministic character of the system, that is the uniqueness of its trajectory between any two events, is achieved by the application of the principle of least action. The application of this principle leads to the trajectory of the system coinciding with the geodesics of the space of reference:

$$\frac{d^2Q^i(s)}{(ds)^2} + \Gamma^i_{jk}\frac{dQ^j(s)}{ds}\frac{dQ^k(s)}{ds} = 0 : \ \Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mj}}{\partial Q^k} + \frac{\partial g_{mk}}{\partial Q^j} - \frac{\partial g_{ij}}{\partial Q^m}\right) \tag{1}$$

where Γ^i_{jk} are the components of the Christoffel symbol of the second kind. The transformation between the arc length s and the time t in the trajectory deduces the equation of motion of the system:

$$\frac{d^2Q^i(t)}{(dt)^2} + \Gamma^i_{jk}\frac{dQ^j(t)}{dt}\frac{dQ^k(t)}{dt} = 0 \tag{2}$$

describing its temporal behavior. The spatiotemporal character of the system is given by the densities $Q^i(\boldsymbol{x})$ of the components of the base, defined as:

$$Q^i(x^0) = \int_{\Phi} Q^i(\boldsymbol{x})d\Phi : d\Phi = dx^1dx^2dx^3 \tag{3}$$

so that the equation of distribution is derived as follows:

$$\sum_{\mu=0}^{3}\left(\frac{\partial^2Q^i(\boldsymbol{x})}{\partial x^\mu \partial x^\mu} + \Gamma^i_{jk}\frac{\partial Q^j(\boldsymbol{x})}{\partial x^\mu}\frac{\partial Q^k(\boldsymbol{x})}{\partial x^\mu}\right) = 0 \Rightarrow \Delta Q^i(\boldsymbol{x}) + \sum_{\mu=0}^{3}\left(\Gamma^i_{jk}\frac{\partial Q^j(\boldsymbol{x})}{\partial x^\mu}\frac{\partial Q^k(\boldsymbol{x})}{\partial x^\mu}\right) = 0 \tag{4}$$

where the Laplace operator of relation 4 is, in general, four-dimensional (spaciotemporal).

Relations 2 and 4 are the constitutional equations of the system, fully describing its temporal evolution and spatial expansion, as functions of the components of the metric tensor of the space of reference (via the components of the Christoffel symbol of the second kind). The application of the third axiomatic principle, namely the principle of Equivalence, determines exactly the metric tensor by connecting it to the external influences (forces) acting upon the system in such a manner that in the case of an inertial system, where no external force or interactions between the components of the system exist, then the space of reference is a Euclidean N-dimensional space. Hence, in the case of an inertial system, a base of the reference space can always be found, namely the usual base $\boldsymbol{U} = \{U^1, \dots, U^N\}$, such that the components of the metric tensor, when referred to the usual base, are $g_{ij} = \delta_{ij}$ and the constitutional equations take the form:

$$\frac{d^2U^i(t)}{(dt)^2} = 0\,, \Delta U^i(\boldsymbol{x}) = 0 \tag{5}$$

The constitutional equations of a general (not necessarily inertial) system of relations 2 and 4 can be restated as follows:

$$\frac{d^2U^i(t)}{(dt)^2} = F^i(t) : \ F^i(t) = -\Gamma^i_{jk}\frac{dU^j(t)}{dt}\frac{dU^k(t)}{dt} \tag{6}$$

and

$$\Delta U^i(\boldsymbol{x}) = F^i(\boldsymbol{x}) : \ F^i(\boldsymbol{x}) = -\sum_{\mu=0}^{3}\left(\Gamma^i_{jk}\frac{\partial U^j(\boldsymbol{x})}{\partial x^\mu}\frac{\partial U^k(\boldsymbol{x})}{\partial x^\mu}\right) \tag{7}$$

such that:

$$F^i(x^0) = \int_{\Phi} F^i(\boldsymbol{x})d\Phi : d\Phi = dx^1 dx^2 dx^3 \tag{8}$$

where $\boldsymbol{F}(t) = \left(F^1(t), \dots, F^i(t)\right)$ and $\boldsymbol{F}(\boldsymbol{x}) = \left(F^1(\boldsymbol{x}), \dots, F^i(\boldsymbol{x})\right)$ are the force and the stress (force density) respectively, corresponding to the external influence and/or the interactions between the components of the system. Generally, the metric tensor can incorporate both transformation of a base in the same space or alteration of the space itself. The use of the usual base in describing the constitutional equations and the application of the principle of Equivalence ensures that the existence of non-vanishing components of the Christoffel symbols corresponds to dynamic state and not to transformation of coordinates. Throughout this paper, the mass m of the system is considered as $m = 1$, to minimize the use of constants. A comprehensive treatment of the physical and mathematical aspects of the above conclusions can be found in i.e. Gelfand and Fomin (1963), Weinstock (1974), Landau and Lifshitz (1980), Dodson and Poston (1997), Francoise, Nabel and Tsun (Editors) (2006), Itskov (2007), Talman (2007) or Bourles (2019).

The above three principles (space of reference, least action and equivalence) are the necessary and sufficient conditions for the derivation of the constitutional equations of the system, so that the last remaining issue is the calibration of these equations to represent the inner attributes of the components of the system, that is the proper choice of the natural base $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$. The usual and the natural base are connected via a permissible transformation $T: \boldsymbol{U} \to \boldsymbol{Q}$ such that $\left|J_{\boldsymbol{U}\to\boldsymbol{Q}}\right| = 0$, where $J_{\boldsymbol{U}\to\boldsymbol{Q}}$ is the Jacobian matrix of the transformation, so that all the functions of the form $Q^i = Q^i(U^1, \dots, U^N)$, $i = 1, \dots, N$, are differentiable, continuous and bijective throughout its domain.

An example of the derivation of the natural base is given in Elias (2023) and Elias (2024), where the system under examination is purely population (all the components represent populations). When such a system is one-dimensional and inertial, its equation of motion described by its usual base is given by relation 5: $U(t) = At + B$, where A and B are real constants. This linear function of time, although corresponds to the dynamic state of the system (one-dimensional and inertial), does not describe its predominant inner property, namely its exponential evolution. The depiction of this property is achieved by the proper choice of a permissible base $\boldsymbol{Q} = \{Q\}$, derived from the transformation $Q = Q(U) = exp(U)$, so that the equation of motion of the system, when described by its natural base $\boldsymbol{Q}$, becomes: $Q(t) = Cexp(At)$, where A and C are real constants, which is the reduction of the Kolmogorov simultaneous equations to Malthus equation.

2. Constitutional equations – Helmholtz equation

In this paper it is assumed that the total force acting upon each component of the usual base, both from the external environment and from other components of the system, can be adequately approximated by a linear function of the components:

$$F^i(U^1, \dots, U^N) = A^i_j U^j : i, j = 1, \dots, N \tag{9}$$

where A^i_j are real constants. Thus, the constitutional equations, described by the usual base (relations 6 and 7) become:

$$\frac{d^2U^i(t)}{(dt)^2} = A^i_j U^j(t) \tag{10}$$

for the equation of motion and:

$$\Delta U^i(\boldsymbol{x}) = A^i_j U^j(\boldsymbol{x}) \tag{11}$$

for the equation of distribution. It can be noticed that relation 11 is a generalization of the Helmholtz wave equation:

$$\frac{\partial^2 f(x,y,z)}{\partial x \partial x} + \frac{\partial^2 f(x,y,z)}{\partial y \partial y} + \frac{\partial^2 f(x,y,z)}{\partial z \partial z} = Af(x,y,z) \tag{12}$$

where $f = f(x,y,z)$ is an arbitrary function of the spatial coordinates and A is a real constant.

Evidently, the simulation of the internal and external force by a linear expression of the components is only an approximation of the real dynamic state of a system, with the sole purpose to simplify the equations and to facilitate the calculations but, since any Riemannian manifold (such as the space of reference of the system) is by definition locally flat, it can be assumed that the linearization of relation 9 represents the actual behavior of the system, at least for a limited spatiotemporal interval. Furthermore:

- the constitutional equations of relation 4 reduce to simultaneous linear differential equations which always produce analytical and relatively simple general solutions,
- all the three important special states of the system, namely the inertial, isolated and fully dynamic, can be described by a simple adjustment of the matrix $[A^i_j]$ in relations 12 or 13,
- the increase of the dimension of the system does not lead to a corresponding increase of the difficulty in solving the constitutional equations, and
- the most significant disadvantage of the linearity of the constitutional equations is its inability to produce emerging characteristics as the dimension of the system increases, which occurs in some complex biological or social systems, such as chaotic behavior.

The application of a suitable linear transformation from the usual base $\boldsymbol{U} = \{U^1, \dots, U^N\}$ to an intermediate base $\boldsymbol{P} = \{P^1, \dots, P^N\}$, such that $U^i = U^i(P^1, \dots, P^N) = B^i_j P^j$, where all the components

B^i_j of the Jacobian matrix are real constants, such that $\left|B^i_j\right| \neq 0$, leads to a simplification of relations 10 and 11:

$$U^i(t) = B^i_j P^j(t) : \frac{d^2 P^j(t)}{(dt)^2} = \lambda^j P^j(t) \tag{13}$$

and

$$U^i(\boldsymbol{x}) = B^i_j P^j(\boldsymbol{x}) : \Delta P^j(\boldsymbol{x}) = \lambda^j P^j(\boldsymbol{x}) \tag{14}$$

The quantities $\lambda^i, i = 1, \ldots, N$ are the eigenvalues of the matrix $\left[A^i_k\right]$ and can take N distinct real or complex constant values. As mentioned in section 1, the transformation between the natural base of the system (where its dynamic state is defined) to its natural base (where its physical attributes are expressed) is achieved by a continues bijective function $\boldsymbol{\mathcal{T}} = \{\mathcal{T}^1, \ldots, \mathcal{T}^N\}$, such that $Q^i\left(U^i\right) = \mathcal{T}^i\left(U^i\right)$. In the special case where (some or all) the components of the system represent populations or follow the exponential growth in their inertial state, the constitutional equations expressed in the natural base are given as:

$$Q^i(t) = \mathcal{T}^i\left(\sum_{j=1}^{N}\left(B^i_j P^j(t)\right)\right) \rightarrow Q^i(t) = exp\left(\sum_{j=1}^{N}\left(B^i_j P^j(t)\right)\right), i = 1, \ldots, N \tag{15}$$

for the equation of motion and:

$$Q^i(\boldsymbol{x}) = \mathcal{T}^i\left(\sum_{j=1}^{N}\left(B^i_j P^j(\boldsymbol{x})\right)\right) \rightarrow Q^i(\boldsymbol{x}) = exp\left(\sum_{j=1}^{N}\left(B^i_j P^j(\boldsymbol{x})\right)\right), i = 1, \ldots, N \tag{16}$$

for the equation of distribution, where the functions $P^j(t)$ and $P^j(\boldsymbol{x})$ are solutions of relations 13 and 14 respectively. Another permissible transformation satisfying the inertial exponential growth can be applied to the natural base $\boldsymbol{Q} = \{Q^1, \ldots, Q^N\}$ in the form of a linear combinations: $\tilde{Q}^i(\boldsymbol{Q}) = C^i_j Q^j + C^i_0$, where C^i_j and C^i_0 are real constants. Since this last transformation retains the properties of the space of reference of the system (mainly its curvature) and the characteristic exponential growth, any base $\tilde{\boldsymbol{Q}} = \left\{\tilde{Q}^1, \ldots, \tilde{Q}^N\right\}$ can be also considered as a natural base of the system.

The solution of the linear ordinary differential equations of relation 13 for different values of the eigenvalues are given by:

$$\begin{gathered} \lambda^i = 0 \Rightarrow P^i(t) = C^i_1 t + C^i_2 \\ \lambda^i \neq 0 \Rightarrow P^i(t) = C^i_1 exp\left(t\sqrt{\lambda^i}\right) + C^i_2 exp\left(-t\sqrt{\lambda^i}\right) \end{gathered} \tag{17}$$

where C_i are real constants and λ^i are constants, either real or complex and the Euler's formula is applied to relation 19, when appropriate. The introduction of complex eigenvalues, that is of a pair of complex conjugates: $\lambda^i = \mu^i + I\nu^i$ and $\lambda^{i+1} = \tilde{\lambda}^i = \mu^i - I\nu^i$, to relation 19 and, finally to the equations of motion and distribution, can be simplified by considering $\sqrt{\lambda^i} = \alpha^i + I\beta^i$, where μ^i, ν^i, α^i and β^i are real constants, such that $\mu^i = \left(\alpha^i\right)^2 - \left(\beta^i\right)^2$, $\nu^i = 2\alpha^i\beta^i$ and $I = \sqrt{-1}$ symbolizes the imaginary unit, to avoid the confusion with the index i. Therefore, a pair of conjugate components of $P^i(t)$ is simplified to:

$$\begin{gathered}\mu^i > 0 \Rightarrow \begin{cases} P^i(t) = C_1^i exp\left(\left(\alpha^i + I\beta^i\right)t\right) + C_2^i exp\left(\left(-\alpha^i - I\beta^i\right)t\right) \\ P^{i+1}(t) = C_1^{i+1} exp\left(\left(-\alpha^i + I\beta^i\right)t\right) + C_2^{i+1} exp\left(\left(\alpha^i - I\beta^i\right)t\right) \end{cases} \\ \mu^i < 0 \Rightarrow \begin{cases} P^i(t) = C_1^i exp\left(\left(-\beta^i + I\alpha^i\right)t\right) + C_2^i exp\left(\left(\beta^i - I\alpha^i\right)t\right) \\ P^{i+1}(t) = C_1^{i+1} exp\left(\left(-\beta^i - I\alpha^i\right)t\right) + C_2^{i+1} exp\left(\left(\beta^i + I\alpha^i\right)t\right) \end{cases}\end{gathered} \tag{18}$$

For a general solution of the elliptic partial differential equations of relation 14 two boundary conditions will apply, the first being that the habitat of the system is a three-dimensional Euclidean space composed by the time and the two-dimensional geographic space, so that $Q^i(x^0, x^1, x^2, x^3) \to Q^i(t, x, y)$ or $Q^i(x^0, x^1, x^2, x^3) \to Q^i(t, \rho, \theta)$, which implies that any solution of relation 14 is of the form that $P^i(t, x, y)$ or $P^i(t, \rho, \theta)$. The second condition deals with the homogeneous and isotropic character of the (geographical) habitat of the system. In the present paper the solutions of the constitutional equations have no favored directions or initial points. The treatment of the geographical diversifications of the habitat (anisotropic and/or inhomogeneous behaviors) was presented in the section 4 of Elias (2024) with the application of Laplace – Beltrami equation. Hence, the Laplace operator can be expressed as:

$$\Delta P^i(t, x, y) = \frac{\partial^2 P^i(t, x, y)}{\partial t \partial t} + \frac{\partial^2 P^i(t, x, y)}{\partial x \partial x} + \frac{\partial^2 P^i(t, x, y)}{\partial y \partial y} \tag{19}$$

where x and y are the longitudinal and latitudinal geographical coordinates respectively, or as:

$$\Delta P^i(t, \rho, \theta) = \frac{\partial^2 P^i(t, \rho, \theta)}{\partial t \partial t} + \frac{\partial^2 P^i(t, \rho, \theta)}{\partial \rho \partial \rho} + \frac{1}{(\rho)^2} \frac{\partial^2 P^i(t, \rho, \theta)}{\partial \theta \partial \theta} + \frac{1}{\rho} \frac{\partial P^i(t, \rho, \theta)}{\partial \rho} \tag{20}$$

where ρ is the radius of the hypothetical center of the system and θ is the azimuthal angle. For the purpose to facilitate the satisfaction of the boundary conditions of the cases of section 3, the Laplace operator of relation 22 is used. Extensive treatment of both Laplace and Helmholtz equations, deriving to sets of analytical solutions, can be found, for example, in Smirnov (1964), Gockenbach (2002), Polyanin (2002), Asmar (2005), Pinchover and Rubinstein (2005), Powers (2006), Drabek and Holubova (2007), Strauss (2008), Taylor (2011) or Sauvigny (2012).

An interesting special case is that of an isolated system, in which the external influences are considered negligent compared to the internal interactions between the components of the system. Since this situation exists in a state of internal dynamic equilibrium, the sum of the forces in relation 9 must me zero for every value of the components $U^i(t)$ or $U^i(\boldsymbol{x})$:

$$\sum_{i=1}^{N}(F^i) = \sum_{j=1}^{N}\left(U^j \sum_{i=1}^{N}(A_j^i)\right) = 0 \Rightarrow \sum_{i=1}^{N}(A_j^i) = 0 \; \forall \, j = 1, \dots, N \tag{21}$$

This last relation leads to the matrix $[A_j^i]$ of relations 12 and 13 to be singular ($|A_j^i| = 0$) and, consequently, at least one eigenvalue λ must be zeroed. It should be noticed that, in contrast to the isolated systems (defined in relation 21), the inertial ones are described as:

$$F^1 = \cdots = F^N = 0 \Rightarrow A_j^i = 0 \; \forall \, i, j = 1, \dots, N \tag{22}$$

and their behavior was presented in Elias (2023).

3. The equation of motion of a Pray – Predator system

It should be emphasized that the main purpose of this paper is not the analysis of Lotka – Volterra equations:

$$\begin{aligned} \frac{dQ^1(t)}{dt} &= C_1^1 Q^1(t) + C_2^1 Q^1(t) Q^2(t) \\ \frac{dQ^2(t)}{dt} &= C_1^2 Q^2(t) + C_2^2 Q^1(t) Q^2(t) \end{aligned} \tag{23}$$

or of prey – predator models specifically, since the starting point of almost all those models involve non – linear dynamics, that is, non – linear differential equations having, in most cases, no analytically derived general solutions. The description and analysis of such models can be found in Lotka (1925), Holling (1959), Keyfitz and Flieger (1971), May (1974), Hoppensteadt and Hyman (1977), Freedman (1980), Gleick (1987), Brauer and Castillo-Chavez (2000), Kot (2001), Galbraith (2006) and in many others. The author's objective is further investigation of the model presented in Elias (2023), leading to constitutional equations that can incorporate, within the same framework, both equations of motion and distribution, easily expandable to N-dimensional systems and (as a personal preference) susceptible to analytical general solutions.

The application of relations 15 and 16 to a two-dimensional system provides the opportunity of a theoretical experiment since, by using a broad definition and a proper choice of the matrix $[A_j^i]$ in relation 9, this system can be considered as approximating some prey-predator behaviors, presented both in the references mentioned in the previous paragraphs and in many relative articles such as Dixon and Cornwell (1970), Berryman (1992), Venkatesha, Brunda, Dhanush and Ambresh (2017), Kumar and Raj (2001), Savadogo, Sangare and Ouedraogo (2022), Smith, Venter, Peel, Keith and Somers (2023), Mekonen, Bezabih and Rao (2024) and Zelenchuk and Tsybulin (2024). These behavioral patterns are given either as temporal functions of the components (populations) or as parametric graphs and are mostly derived from numerical processes or from analyzing characteristic situations, i.e. equilibrium or stability. These results are comparable to the analytically derived equation of motion (and the consequent parametric graphs) of relation 15, presented in the remaining of this section.

The approach of section 2 can:

- simultaneously derive both the temporal (equation of motion) and the spaciotemporal (equation of distribution) character of the system, the latter being essential for a number of disciplines, including regional science, urban planning, transport optimization, epidemiology or biology,
- provide easily obtainable analytical general solutions, regardless of the dimension of the system,
- describe components of varied nature (population, demographic, economic etc.) with the same constitutional equations, by the application of the proper transformation $\boldsymbol{\mathcal{T}} = \{\mathcal{T}^1, \dots, \mathcal{T}^N\}$.

Evidently, the above capabilities derive from the linearization of the dynamic terms of the constitutional equations (see relations 6 and 9), hence cannot reach the complexity, analytical depth and the emerging behavior of the non - linear prey – predator models derived as variations of relation 23, especially when dealing with systems of higher dimensions and chaotic behavior. There two issues worth further investigation: a) the set of situations in which relation 15 can well-approximate the non-linear models and, more importantly, b) the set of situations in which relation 15 can well-approximate naturally occurring systems (at least as well as the non-linear models can).

In using the constitutional equations of section 2 as a pray – predator system (or an approximation of it) , its inertial state is achieved when all the components of the matrix $[A_j^i]$ in relations 10 and 11 vanish: $A_j^i = 0 \; \forall \; i, j = 1, \dots, N$, in which case the equation of motion take its simplest exponential form: $Q^i(t) = exp\left(C_1^i t + C_2^i\right)$. The isolated (dynamic equilibrium) state, where the external forces are negligent compared to the interaction between the components (populations), is derived by imposing the constraint of relation 21 to the constitutional equations in relations 10 and 11. The main features (matrices and eigenvalues) can be calculated as:

$$\boldsymbol{A} = [A_k^i] = \begin{bmatrix} A_1^1 & A_2^1 \\ -A_1^1 & -A_2^1 \end{bmatrix}, \boldsymbol{\lambda} = [\lambda^i] = \begin{bmatrix} 0 \\ A_1^1 - A_2^1 \end{bmatrix}, \boldsymbol{B} = [B_k^i] = \begin{bmatrix} -A_2^1/A_1^1 & -1 \\ 1 & 1 \end{bmatrix} \tag{24}$$

where all the components of the above matrices and vectors are real constants. Hence, the general equation of motion of an isolated system becomes:

$$\begin{gathered} Q^1(t) = exp\left(-(A_2^1/A_1^1)P^1(t) - P^2(t)\right) \\ Q^2(t) = exp\left(P^1(t) + P^2(t)\right) \end{gathered} \tag{25}$$

where $P^1(t)$ corresponds to the eigenvalue $\lambda^1 = 0$ and $P^2(t)$ to $\lambda^2 = A_1^1 - A_2^1 \neq 0$ and their analytical expressions (see relations 17) are:

$$\lambda^2 = A_1^1 - A_2^1 < 0$$

$$Q^1(t) = exp\left(-(A_2^1/A_1^1)C_1^1 t - \frac{A_2^1}{A_1^1}C_2^1 - C_1^2 cos\left(t\sqrt{|\lambda^2|}\right) - C_2^2 sin\left(t\sqrt{|\lambda^2|}\right)\right) \quad (26)$$

$$Q^2(t) = exp\left(C_1^1 t + C_2^1 + C_1^2 cos\left(t\sqrt{|\lambda^2|}\right) + C_2^2 sin\left(t\sqrt{|\lambda^2|}\right)\right)$$

or:

$$\lambda^2 = A_1^1 - A_2^1 > 0$$

$$Q^1(t) = exp\left(-(A_2^1/A_1^1)C_1^1 t - \frac{A_2^1}{A_1^1}C_2^1 - C_1^2 exp\left(t\sqrt{\lambda^2}\right) - C_2^2 exp\left(-t\sqrt{\lambda^2}\right)\right) \quad (27)$$

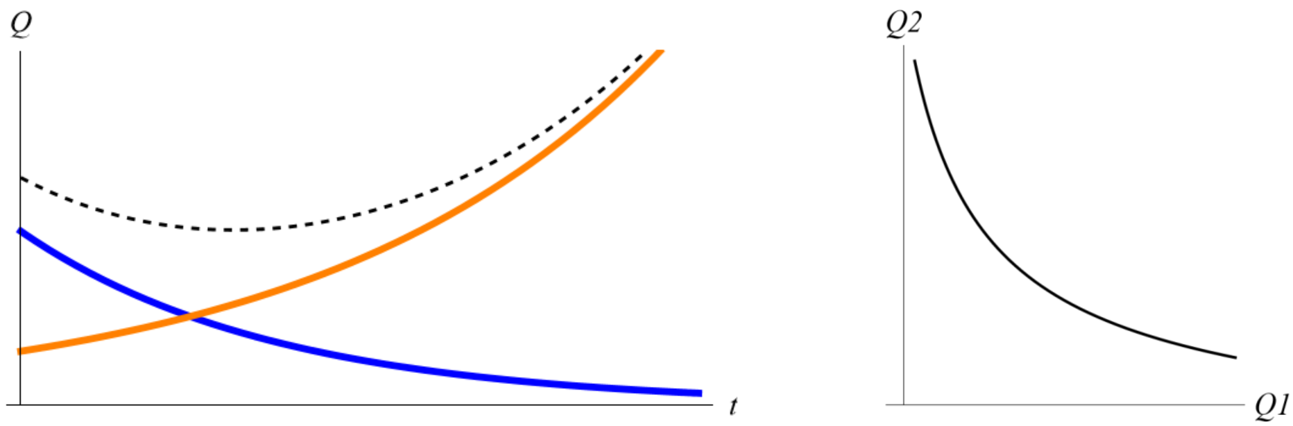
$$Q^2(t) = exp\left(C_1^1 t + C_2^1 + C_1^2 exp\left(t\sqrt{\lambda^2}\right) + C_2^2 exp\left(-t\sqrt{\lambda^2}\right)\right)$$

where C_j^i are real constants.

Relations 26 and 27 produce four main temporal behaviors for isolated systems, depicted by Figures 1 to 4, each of which includes some variations, depending on the values of the real constants A_j^i and C_j^i. The simplest pattern is derived by considering: $C_1^2 = C_2^2 = 0$ (in either relations 26 or 27), representing an inertial system, since only the component $P^1(t)$ contributes to the equation of motion. and is depicted in Figure 1. The second pattern occurs by eliminating the constant C_1^1 ($C_1^1 = 0$) from relation 26, resulting in purely periodic functions of time for both components of equation of motion and, thus, describing a sustainable (symbiotic) isolated system, as depicted in Figure 2. The existence, or even the pursuit of achieving, such a symbiosis is of profound importance for some key systems.

Figure 1. Representation of relations 26 or 27, where $C_1^2 = C_2^2 = 0$. a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 1.a.

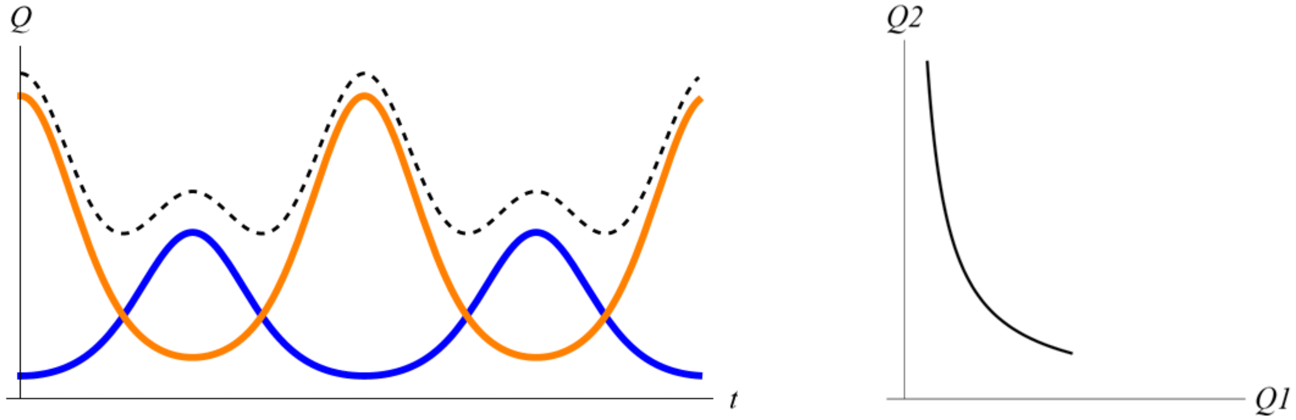
(a) (b)



Source: Author's representation

Figure 2. Representation of relation 26 in the case of $C_1^1 = 0$. a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 2.a.

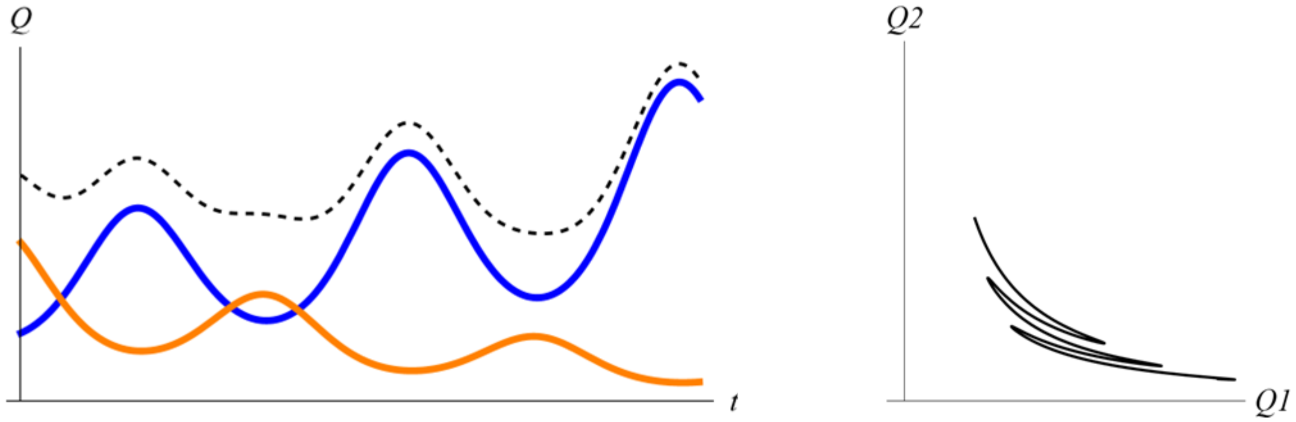
(a) (b)



Source: Author's representation

Figure 3. Representation of relation 27 in the case of $C_j^i \neq 0$. a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 2.a.

(a) (b)



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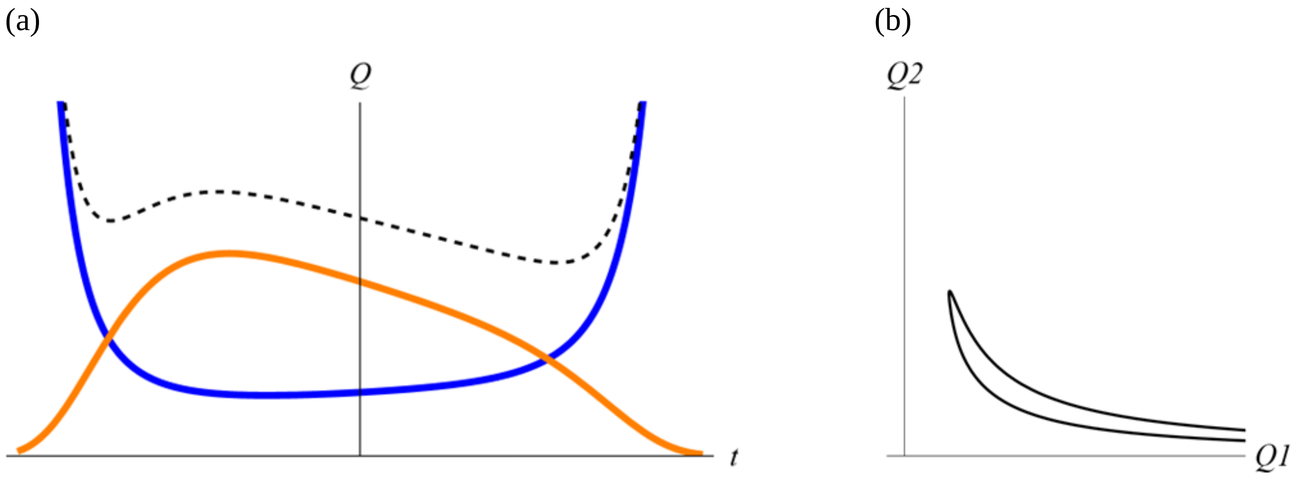
The complete character of relation 26, with the participation of all its terms, is depicted in Figure 3, where $C_j^i \neq 0$. This pattern includes both the internal conflict (balance) among the components, manifested by the trigonometric terms of the equation and, also, the eventual dominance of one component, due to the difference of the constants of the linear terms (finally, the exponential terms). The final pattern, depicted in Figure 4, is derived from relation 27, with the participation of all the constants ($C_j^i \neq 0$) and describes an aggressively competitive (conflicting) system, which behavior is the complete opposite to that depicted in Figure 2. Indeed, in this case, one component leads eventually the other to virtual elimination and, consequently, to the collapse of the whole system. Evidently, the constitutional equations are valid during the existence of the system, that is, as long as there is no eliminated component. For example, in Figure 4.a and Figure 6.a, after an interactive period the components Q^2 (orange lines) become so small that they can be considered as negligible, hence, from that time on, the systems collapse and the components Q^1 behave as inertial one-dimensional systems (following purely exponential growth).

Figure 4. Representation of relation 28 in the case of $C_j^i \neq 0$. a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 4.a.



Source: Author's representation

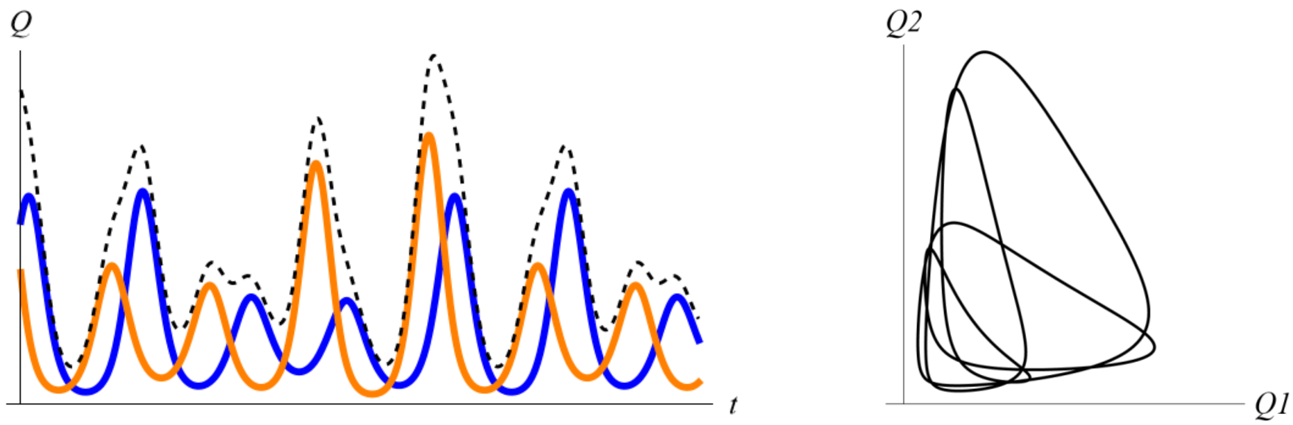
In a dynamic system affected by external influences, the real constants B_j^i are not subject to the restrictions of relation 21, therefore its temporal behavior patterns are more complex than that of the isolated case and the equation of motion takes the following general form of relation 15:

$$Q^i(t) = exp\left(B_1^i P^1(t) + B_2^i P^2(t)\right) : i = 1,2 \tag{29}$$

where $P^i(t)$ are given in relation 17, B_j^i and C_j^i are real constants and the eigenvalues $\lambda^i \neq 0$ can be real or complex constants. In a two-dimensional system, the complex eigenvalues include only the case of a single pair of complex conjugates eigenvalues, hence the system cannot be isolated. The general equation of motion of relation 29 is applied, where the functions $P^i(t)$ are taken from relation 18 and is represented in Figure 8. By comparing similar cases of isolated and dynamic systems, the effect of external influences becomes apparent, as in Figures 2 and 5, for symbiotic (sustainable) systems, Figures 4, 6 and 7.c, for competitive (adversarial) systems or Figures 1, 3 and 8.

Figure 5. Representation of relation 29 in the case of $\lambda^1 < 0, \lambda^2 < 0$ (symbiotic system). a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 5.a.

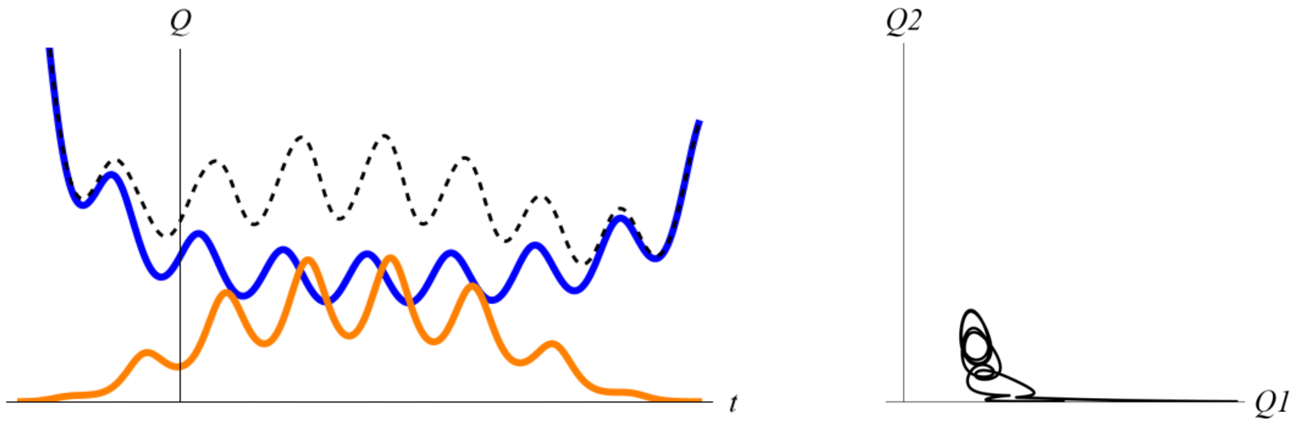
(a) (b)



Source: Author’s representation

Figure 6. Representation of relation 29 in the case of $\lambda^1 < 0$, $\lambda^2 > 0$ (competitive system). a) Temporal behavior where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 6.a.

(a) (b)

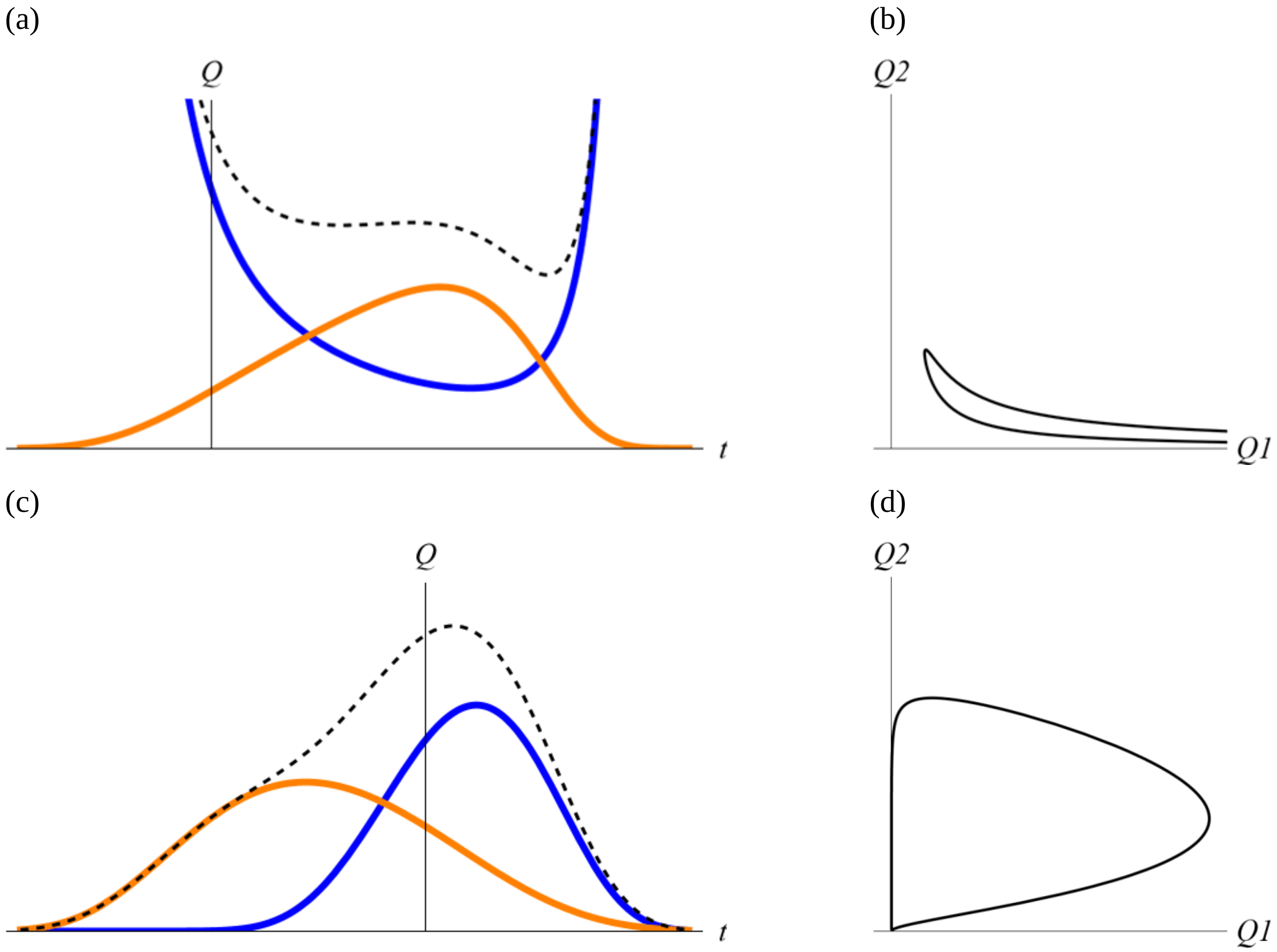


Source: Author’s representation

Some applications of this section can be illustrated by two simplistic examples concerning competitive systems (Figures 4, 6 and 7):

- Epidemiology: Starting with Figure 4, let Q^2 (orange line) and Q^1 (blue line) represent the infected and healthy population respectively of an epidemic situation, then the temporal functions correspond to the natural evolution of the disease (initial infection, outbreak and weakening). To minimize both the duration of the epidemic and the maximum number of infected people, society takes additional measures, acting as external force of the system such as vaccinations, isolation and hospitalization, which are manifested as fluctuations of the temporal functions of the populations (Figure 6).

Figure 7. Representation of relation 29 in the case of $\lambda^1 > 0$, $\lambda^2 > 0$ for different values of the constants B_j^i. a) Temporal behavior of competitive components, where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b) Parametric plot of Figure 6.a. c) Temporal behavior of coexisting components, where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. d) Parametric plot of Figure 6.c.

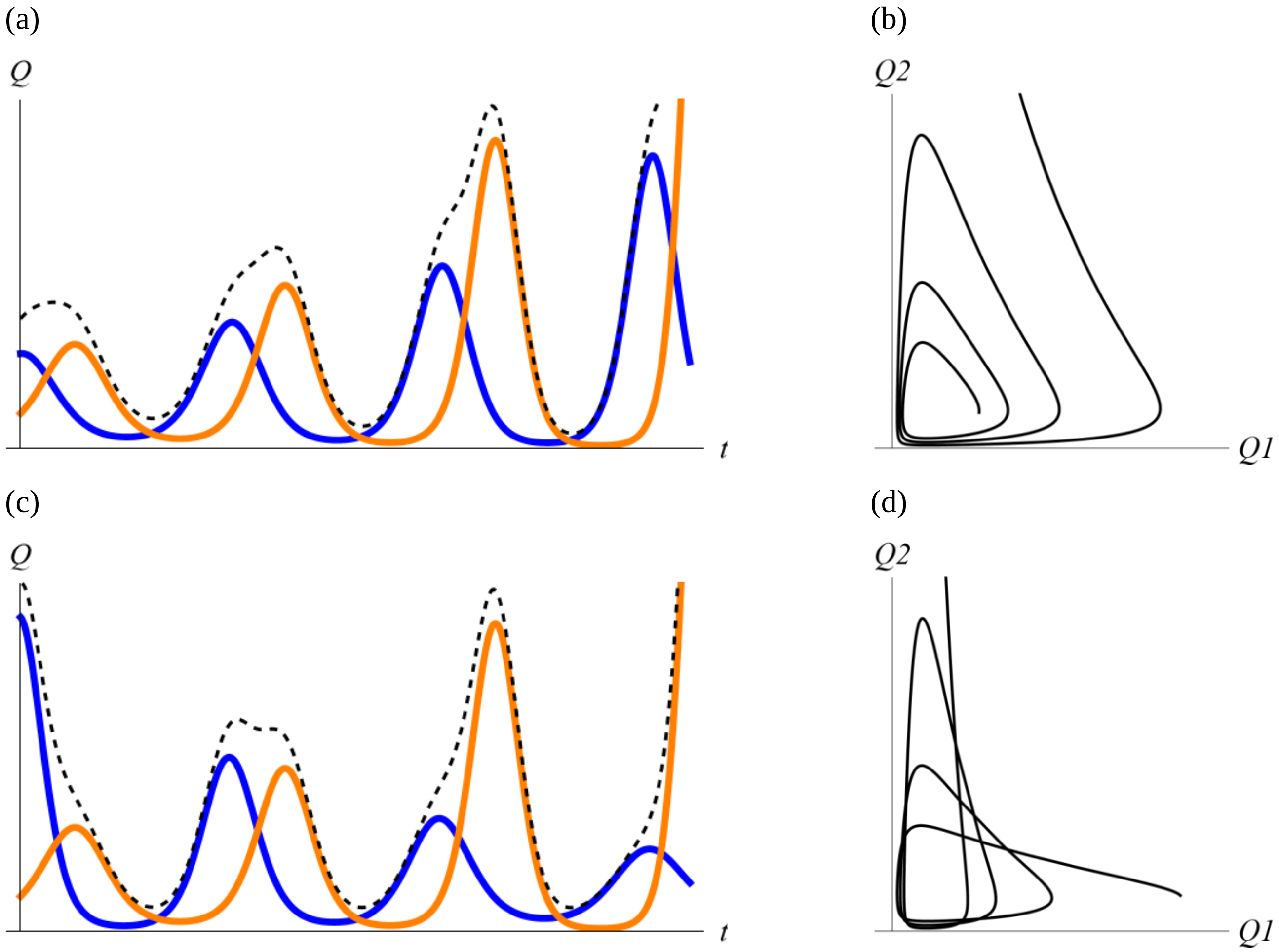


Source: Author’s representation

- History: Let Q^2 be a structured and compact society (say Rome) which expands at the expense of the neighboring societies (Q^1). In Figure 4, the classical scheme is demonstrated (rise, heyday and decline of Q^2) with the corresponding decline and rise of Q^1, on the assumption that no major changes occur (acting as external forces) in both societies. The major structuring changes (Marious reform, transformation from republic to empire, Diocletian reform, destruction of the middle class etc.) of Rome and the corresponding improved organization and the migration (from the East) of the surrounding societies, led to the fluctuations of Figure 6 and/or the transfer of the heyday of Rome by some centuries (Figure 7.a), The final fall of Rome and the creation of its successors (Byzantium and Holy Roman empire) is represented in Figure 7.c.

As far as the historical examples are concerned, the decrease of the population does not necessarily represent its biological extermination but, rather, the gradual detachment of a large group of people from the main body of society, that is, of people no longer interacting with the system and, hence, being irrelevant. Please keep in mind that the purpose of the above crude examples is only to give some perspective to the equations and their accompanying Figures and should not be interpreted as some expert analysis on Biology, Sociology or History.

Figure 8. Representation of relation 29 in the case that λ^1 and λ^2 are a pair of complex conjugates, for different values of the constants B_j^i and C_j^i. a and c: Temporal behavior of the components where Q^1, Q^2 and $Q^1 + Q^2$ are depicted by blue, orange and dashed black lines respectively. b and d: Parametric plots of a and c respectively.



Source: Author’s representation

4. The equation of distribution of a Pray – Predator system

The equation of distribution of a pray-predator system is derived as a special case of relation 16 for $N = 2$, where $P^i(\boldsymbol{x})$ are solutions of relation 14, where, for reasons of convenience, the components $Q^i(\boldsymbol{x})$ and the building blocks $P^i(\boldsymbol{x})$ are expressed in cylindrical coordinates of the spatiotemporal habitat of the system, that is: $Q^i = Q^i(t, \rho, \theta)$ and $P^i = P^i(t, \rho, \theta)$ and the Laplace operator takes the form of relation 20. Therefore, by applying relations 16 and 24, the final expression of the equation of distribution becomes:

$$Q^1(t, \rho, \theta) = exp\big(-(A_2^1/A_1^1)P^1(t, \rho, \theta) - P^2(t, \rho, \theta)\big)$$

$$Q^2(t, \rho, \theta) = exp\big(P^1(t, \rho, \theta) + P^2(t, \rho, \theta)\big) \qquad (30)$$

such that: $\Delta P^1(t, \rho, \theta) = 0$ and $\Delta P^2(t, \rho, \theta) = (A_1^1 - A_2^1)P^2(t, \rho, \theta)$

for the isolated (equilibrium) systems and

$$Q^i(t, \rho, \theta) = exp\left(B_1^i P^1(t, \rho, \theta) + B_2^i P^2(t, \rho, \theta)\right) : \ \Delta P^i(t, \rho, \theta) = \lambda^i P^i(t, \rho, \theta) \qquad (31)$$

for dynamic systems, where A_j^i and B_j^i are real constants and the eigenvalues λ^i can be real or complex (pairs of complex conjugates) constants, with $i = 1,2$.

All the components of the system (relations 30 and 31) are subject to the following boundary conditions:

- The domain of the functions $Q^i(t, \rho, \theta)$ is $t \in]-\infty, +\infty[$, $\rho \in [0, +\infty[$ and $\theta \in [0,2\pi]$ and their range is $Q^i \in [0, +\infty[$.
- All $Q^i(t, \rho, \theta)$ tend to zero as ρ tends to infinity, for all values of t and θ. This ensures that the geographical area of the habitat may be time-dependent but not limitless.
- All $Q^i(t, \rho, \theta)$ depend on some periodic periodic function of θ or are independent from it.
- For every finite value of t, all $Q^i(t, \rho, \theta)$ take finite values for all values of ρ and θ. In some cases of isolated systems (relation 30), there is one point of the habitat, namely $\rho = 0$, where one of the components of the population density appears to take infinite value, which means that, at that point, the density cannot be determined. For dynamic systems this irregularity can be eliminated.

Three examples of equation of distribution are presented in this section, namely the isolated symbiotic system of Figure 2 and the simplified versions of the dynamic symbiotic and competitive systems of Figures 5 and 7 respectively.

The object of this section is not to provide a complete catalogue of the general solutions of relations 30 or 31, since that would be too extensive and beyond the scope of this paper, but to present and examine some interesting particular solutions of population distribution, especially those having non-homogeneous and non-isotropic behavior and, simultaneously, obeying the above boundary conditions. Concerning the example of the isolated symbiotic system of Figure 2, its equation of motion of relation 27 contains a function $P^1(t) = C_2^1$, which is independent of time, hence its equation of distribution should take the following form:

$$\begin{aligned} Q^1(t,\rho,\theta) &= exp\left(-(A_2^1/A_1^1)P^1(\rho) - P^2(t,\rho,\theta)\right) \\ Q^2(t,\rho,\theta) &= exp\left(P^1(\rho) + P^2(t,\rho,\theta)\right) \end{aligned} \tag{33}$$

where the proposed solutions of $P^1(t,\rho,\theta)$ and $P^2(t,\rho,\theta)$ are derived by Laplace and Helmholtz equation respectively:

$$\begin{aligned} &\lambda^1 = 0 \Rightarrow P^1(t,\rho,\theta) = P^1(\rho) = C_1^1 ln(\rho) + C_2^1 \\ &\lambda^2 < 0 \Rightarrow P^2(t,\rho,\theta) \\ &= \left(C_1^2 cos\left(t\sqrt{|\lambda^2|}\right) + C_2^2 sin\left(t\sqrt{|\lambda^2|}\right)\right) \\ &+ \sum_{K>0}^{+\infty}\sum_{n=0}^{+\infty}\left(\left(C_{1,Kn}^2 cos\left(t\sqrt{|\lambda^2 - K|}\right)\right.\right. \\ &\left. + C_{2,Kn}^2 sin\left(t\sqrt{|\lambda^2 - K|}\right)\right)\left(C_{3,Kn}^2 J_n\left(\rho\sqrt{K}\right)\right. \\ &\left.\left. + C_{4,Kn}^2 Y_n\left(\rho\sqrt{K}\right)\right)\left(C_{5,Kn}^2 cos(n\theta) + C_{6,Kn}^2 sin(n\theta)\right)\right) \end{aligned} \tag{34}$$

where C_j^i are real constants, K is non-negative real constant, n is non-negative integer constant and $J_n(\rho\sqrt{K})$ and $Y_n(\rho\sqrt{K})$ are the Bessel functions of the first and second kind for order n respectively.

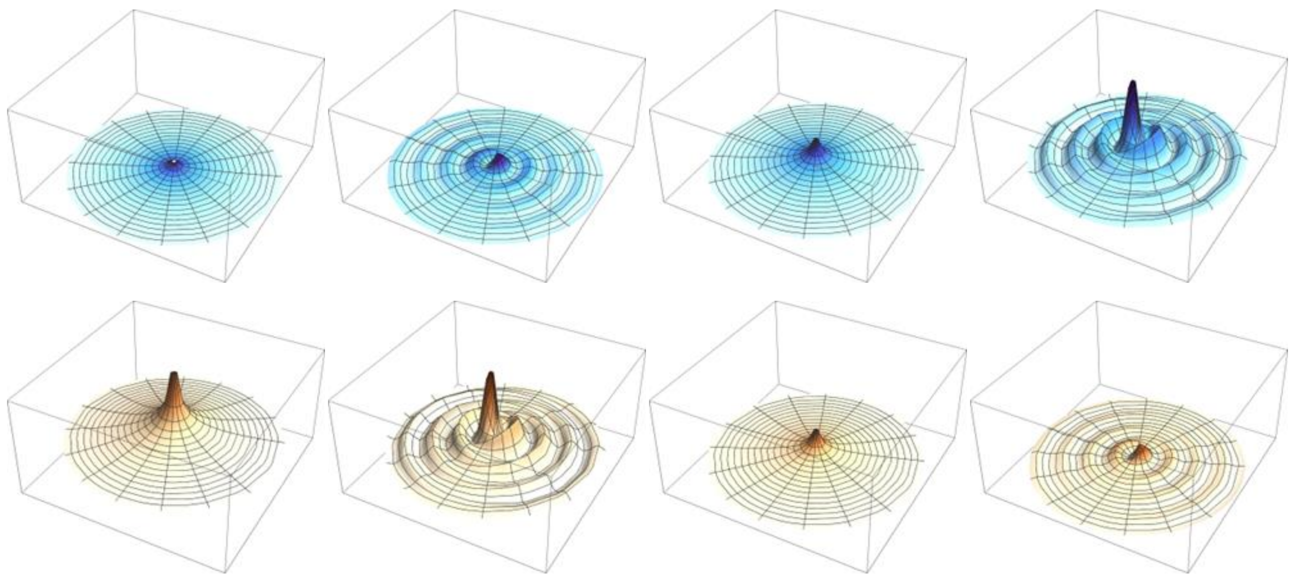
In relation 34 (as combined by relation 33) three different modules can be detected which serve equinumerous operations: a) the function $P^1(\rho)$ which ensures the existence of a finite habitat of the system, b) the first two terms of the function $P^2(t,\rho,\theta)$ (outside the double sum) that regulates the equation of distributions to be consistent to the equation of motion (Figure 2) and c) the double sum of the function $P^2(t,\rho,\theta)$ which regulates the manner of the distribution of the populations (for example, if $n = 0$ or $C_{j,Kn}^2 = 0\ \forall j$ the equation of distribution becomes purely isotropic for all instants of time). The simplest non-isotropic equation of distribution is achieved by substituting the constants $K = -3\lambda^2$ and $n = 1$ in relation 34, thus reducing function $P^2(t,\rho,\theta)$ to:

$$\begin{aligned} &\lambda^2 < 0, K = -3\lambda^2 \Rightarrow P^2(t,\rho,\theta) \\ &= C_1^2 cos\left(t\sqrt{|\lambda^2|}\right) \\ &+ C_{1,Kn}^2 cos\left(2t\sqrt{|\lambda^2|} + \pi/2\right) C_{3,K1}^2 J_1\left(\rho\sqrt{-3\lambda^2}\right) C_{5,K1}^2 cos(1\theta) \end{aligned} \tag{35}$$

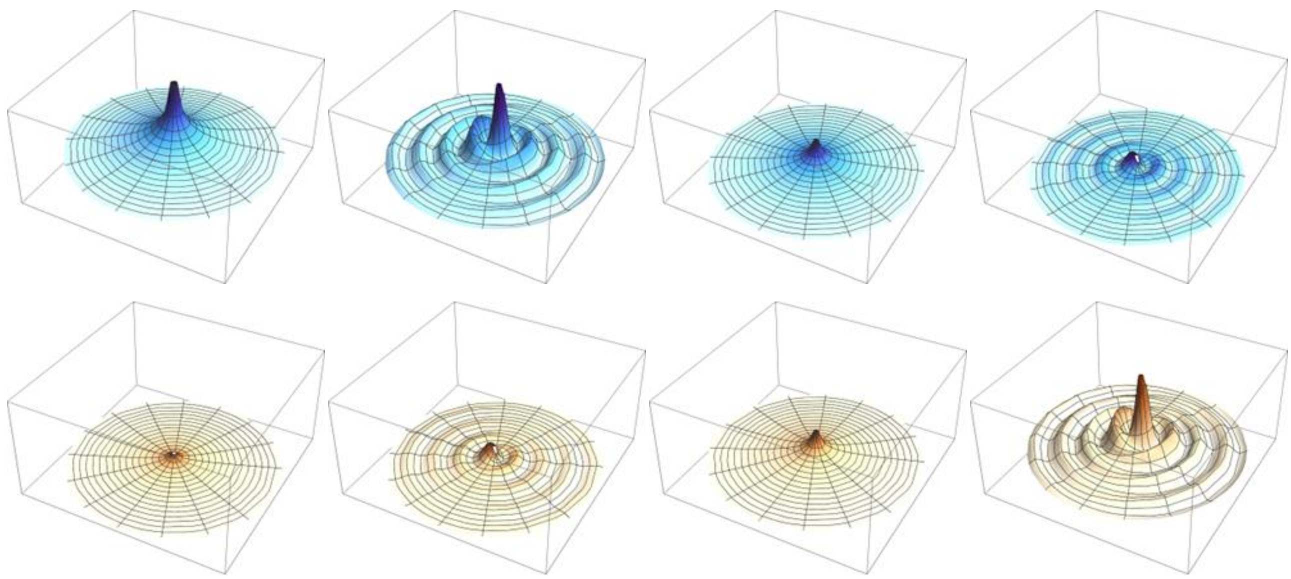
Figure 9 depicts the replacement of relation 35 to 33 for eight instances of time.

Figure 9. Representation of relations 33 and 35 ($K = -3\lambda^2$ and $n = 1$), corresponding to a non-isotropic symbiotic isolated system, where the two populations Q^1 and Q^2 are depicted by blue and orange surfaces respectively with the darker color corresponding to higher density. Figures 9.a to 9.h correspond to eight instances of a complete period T of the history of the system, in accordance with Figure 2 and are all on the same scale.

(a) $t = 0T/8 \equiv 8T/8$ (b) $t = 1T/8$ (c) $t = 2T/8$ (d) $t = 3T/8$



(e) $t = 4T/8$ (f) $t = 5T/8$ (g) $t = 6T/8$ (h) $t = 7T/8$



Source: Author’s representation

The following remarks can be made concerning Figure 9:

- This example is chosen to emphasize the behavior of two populations having the tendency to mostly concentrate in an area near the geometric center of their habitat (city).

- The particular solution of relation 34 produces an equation of distribution that provides a finite habitat, thus obeying the boundary conditions of the system but, for an isolated system, the population density at the exact geometric center of the habitat cannot be defined.
- The distribution of the populations is non-isotropic for all times during the period T except at four instances ($t = 0T/8 \equiv 8T/8$, $t = 2T/8$, $t = 4T/8$, $t = 6T/8$) in Figures 9.a, 9.c, 9.e and 9.g respectively. Between the isotropic instances, the density of both populations decreases or increases in a non-isotropic manner, in which the increase of a population implies the decrease of the other.
- During the non-isotropic distribution, the density center (maximum density) of each population does not coincide with the geometric center of the habitat but, rather, the density centers of the two populations occupy symmetrically opposing points with respect to the geometric center. Furthermore, after every isotropic instant, the density center of a population reverses its orientation and occupies the point of the density center of the other population.
- The continued symmetrical interchange signifies the existence of a dynamic and isolated system and the balanced conflict around the geometric center (the dominant position) permits the system to be symbiotic (sustainable).

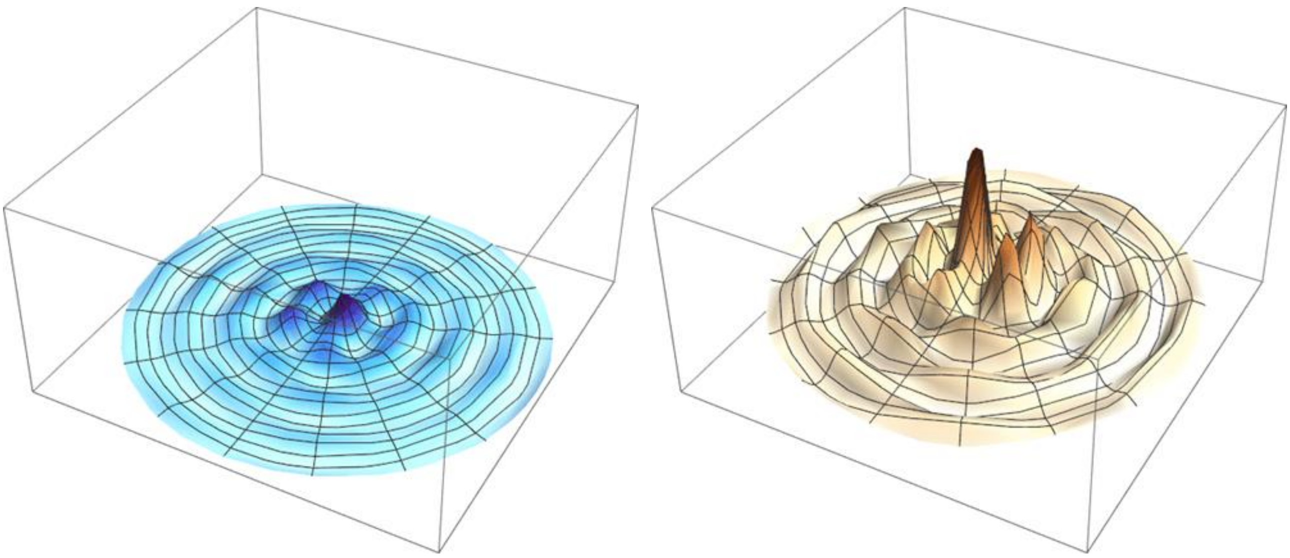
As previously mentioned, Figure 9 (relation 35) represents the simplest non-isotropic distribution derived from relations 33 and 34 but the use of more terms of the double sum of relation 34 quickly leads to much more complex distribution patterns. To illustrate the formation of this complexity, one more term of relation 34 is used, by retaining $K = -3\lambda^2$ and $n = 1$, as in the case of Figure 9 and adding the term corresponding to $n = 5$, thus restate relation 35 as follows:

$$\begin{aligned}\lambda^2 < 0, K = -3\lambda^2 \Rightarrow P^2(t,\rho,\theta) \\ &= C_1^2 cos\left(t\sqrt{|\lambda^2|}\right) \\ &+ C_{1,Kn}^2 cos\left(2t\sqrt{|\lambda^2|} + \pi/2\right)\left(C_{3,K1}^2 J_1\left(\rho\sqrt{-3\lambda^2}\right) C_{5,K1}^2 cos(1\theta)\right. \\ &\left.+ C_{3,K5}^2 J_5\left(\rho\sqrt{-3\lambda^2}\right) C_{5,K5}^2 cos(5\theta)\right)\end{aligned} \tag{36}$$

The representation of the replacement of relation 36 to relation 33 is depicted in Figure 10 for a single instant of time, namely $t = 1T/8$, so that the immediate comparison of Figure 10 to Figure 9.b is possible. Indeed, apart from the central high-density area, the propagation of the populations from the center to the periphery of the habitat (city) is described by Figure 9 as wave-lake high density formations covering almost half the perimeter of the habitat. On the other hand, the additional term in relation 36 describes this propagation in more detail and indicates the existence of high-density population concentrations, forming neighborhoods (or satellite cities), as shown in Figure 10. Nevertheless, the indiscriminate use of many terms, for different values of K and n in the double sum of relation 34 can easily and quickly lead to population distribution that appears random and almost unintelligible, which defeat the effort of detailed description.

Figure 10. Representation of relation 33 and 36, where the two population Q^1 and Q^2 are depicted by blue and orange surfaces respectively with the darked color corresponding to higher density. Figures 10.a and 10.b are on the same scale and are comparable to Figures 9.b.

(a) (b)



Source: Author's representation

The equation of motion of a dynamic symbiotic system is given by relation 29 and Figure 5, with two distinct eigenvalues $\lambda^i < 0$, hence its corresponding equation of distribution is given by relation 31, with both eigenvalues $\lambda^i < 0$. A set of proposed particular solutions of $P^i(t, \rho, \theta)$ are the following:

$$\begin{aligned}\lambda^i < 0 \Rightarrow P^i(t, \rho, \theta) &\\ = C_1^i J_0\left(-I\rho\sqrt{|\lambda^i|}\right) + \left(C_2^i cos\left(t\sqrt{|\lambda^i|}\right) + C_3^i sin\left(t\sqrt{|\lambda^i|}\right)\right) &\\ + \sum_{K^i>0}^{+\infty} \sum_{n^i=0}^{+\infty} \left(\left(C_{1,K^i n^i}^i cos\left(t\sqrt{|\lambda^i - K^i|}\right)\right.\right. &\\ \left. + C_{2,K^i n^i}^i sin\left(t\sqrt{|\lambda^i - K^i|}\right)\right)\left(C_{3,K^i n^i}^i J_{n^i}\left(\rho\sqrt{K^i}\right)\right. &\\ \left.\left. + C_{4,K^i n^i}^i Y_{n^i}\left(\rho\sqrt{K^i}\right)\right)\left(C_{5,K^i n^i}^i cos(n^i\theta) + C_{6,K^i n^i}^i sin(n^i\theta)\right)\right) &\end{aligned} \quad (37)$$

where C_j^i, $K^i > 0$, $n^i > 0$ and $\lambda^i < 0$ are real constants, $J_{n^i}(\rho\sqrt{K^i})$ and $Y_{n^i}(\rho\sqrt{K^i})$ are the Bessel functions of the first and second kind for order n respectively and $J_0(I\rho\sqrt{K^i})$ is the modified Bessel function of the first kind for order 0. There are some significant behavioral differences between relations 34 (and 33) and 37 (and 31), apart from the fact that they describe different dynamic states. Primarily, the module enforcing the finite habitat of an isolated system is: $exp(C_1^1 ln(\rho) + C_2^1)$ where $C_1^1 < 0$, which leads to an equation of distribution that cannot be defined at the geometric center of the habitat ($\rho = 0$), since no one component of a symbiotic system can permanently occupy the strategically dominant position of the geometric center. On the other hand, for a dynamic system the corresponding module: $exp\left(C_1^i J_0\left(I\rho\sqrt{|\lambda^i|}\right)\right)$ where $C_1^1 < 0$, can be used, so that all the components of the equation of distribution can be defined for all the points of the habitat, including its geometric center, hence, in this case, the dominant position can be permanently occupied by one population (component). Secondly, a dynamic system permits a thorough investigation of the interaction (or confrontation) between centrally and peripherally located populations in the same habitat (city or country). The following simplified example demonstrates exactly that interaction.

The second example describes a dynamic adversarial system as represented in Figures 6 and 7, the general equation of distribution of which is given by relation 31, where both eigenvalues are positive real constants ($\lambda^i > 0$). A possible particular solution for the functions $P^i(t,\rho,\theta)$, for positive eigenvalues is presented in relation 38.

$$\begin{aligned}\lambda^i > 0\,, K_0^i < 0 \Rightarrow P^i(t,\rho,\theta)&\\ = C_1^i J_0\left(-I\rho\sqrt{|K_0^i|}\right)&\left(C_2^i exp\left(t\sqrt{\lambda^i - K_0^i}\right) + C_3^i exp\left(-t\sqrt{\lambda^i - K_0^i}\right)\right)\\ &+ \left(C_4^i exp\left(t\sqrt{\lambda^i}\right) + C_5^i exp\left(-t\sqrt{\lambda^i}\right)\right)\\ &+ \sum_{K^i>0}^{\lambda^i}\sum_{n^i=0}^{+\infty}\left(\left(C_{1,K^i n^i}^i exp\left(t\sqrt{\lambda^i - K^i}\right)\right.\right.\\ &+ \left.C_{2,K^i n^i}^i exp\left(-t\sqrt{\lambda^i - K^i}\right)\right)\left(C_{3,K^i n^i}^i J_{n^i}\left(\rho\sqrt{K^i}\right)\right.\\ &+ \left.\left.C_{4,K^i n^i}^i Y_{n^i}\left(\rho\sqrt{K^i}\right)\right)\left(C_{5,K^i n^i}^i cos\left(n^i\theta\right) + C_{6,K^i n^i}^i sin\left(n^i\theta\right)\right)\right)\end{aligned} \tag{38}$$

The application of relation 38 (and of relation 37) to relation 31 can lead to rather complex distribution patterns, so for reasons of simplicity of the depictions, the following example is considered to have a diagonal matrix as given by relations 39:

$$\begin{aligned}Q^i(t) &= exp\left(B_i^i P^i(t)\right)\\ Q^i(t,\rho,\theta) &= exp\left(B_i^i P^i(t,\rho,\theta)\right)\end{aligned} \tag{39}$$

where $i = 1,2$. Furthermore, relation 38 can be simplified to relation 40 by choosing $K^i = \lambda^i$, which still retains the basic distribution characteristics of relation 38:

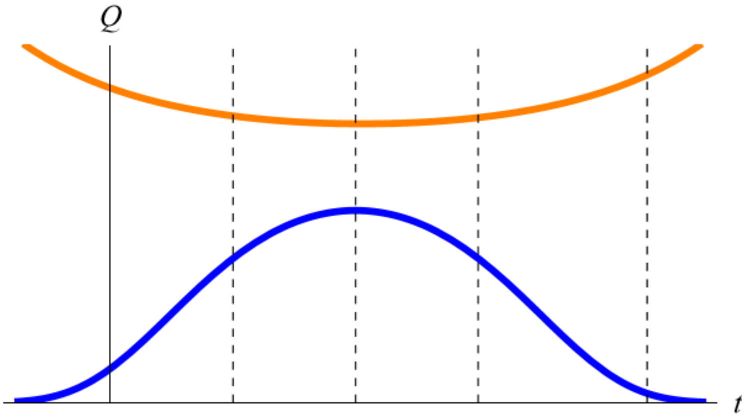
$$\lambda^i > 0\,, K_0^i < 0 \Rightarrow P^i(t,\rho,\theta)$$

$$= C_1^i J_0\left(-I\rho\sqrt{\left|K_0^i\right|}\right)\left(C_2^i exp\left(t\sqrt{\lambda^i - K_0^i}\right) + C_3^i exp\left(-t\sqrt{\lambda^i - K_0^i}\right)\right)$$
$$+ \left(C_4^i exp\left(t\sqrt{\lambda^i}\right) + C_5^i exp\left(-t\sqrt{\lambda^i}\right)\right)$$
$$+ \sum_{n^i=0}^{+\infty}\left(\left(C_{1,n^i}^i J_{n^i}\left(\rho\sqrt{\lambda^i}\right) + C_{2,n^i}^i Y_{n^i}\left(\rho\sqrt{\lambda^i}\right)\right)\left(C_{3,n^i}^i cos(n^i\theta)\right.\right.$$
$$\left.\left. + C_{4,n^i}^i sin(n^i\theta)\right)\right) \tag{40}$$

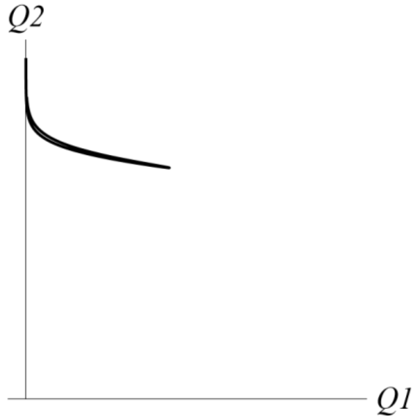
The corresponding equation of motion of the simplified system of relation 39 (see relation 13 for $\lambda^i > 0$) is depicted in Figure 11.

Figure 11: a) Representation of a simplified equation of motion for an adversarial dynamic system in relation 39 where the components Q^1 and Q^2 are depicted by blue and orange lines respectively. The dashed vertical lines indicate the instants ($t_0 > t_1 > t_2 > t_3 > t_4$) for which the equation of distribution is depicted (Figures 12 and 13), where t_0 corresponds to $t_0 = 0$, $t_2 = t_{ext}$ to the instant when the components take their extreme values, instances t_1 and t_3 are such that $Q^i(t_1) = Q^i(t_3)$ and the instant t_4 indicates final stage of Q^1. b) Parametric plot of Figure 11.a.

(a)



(b)

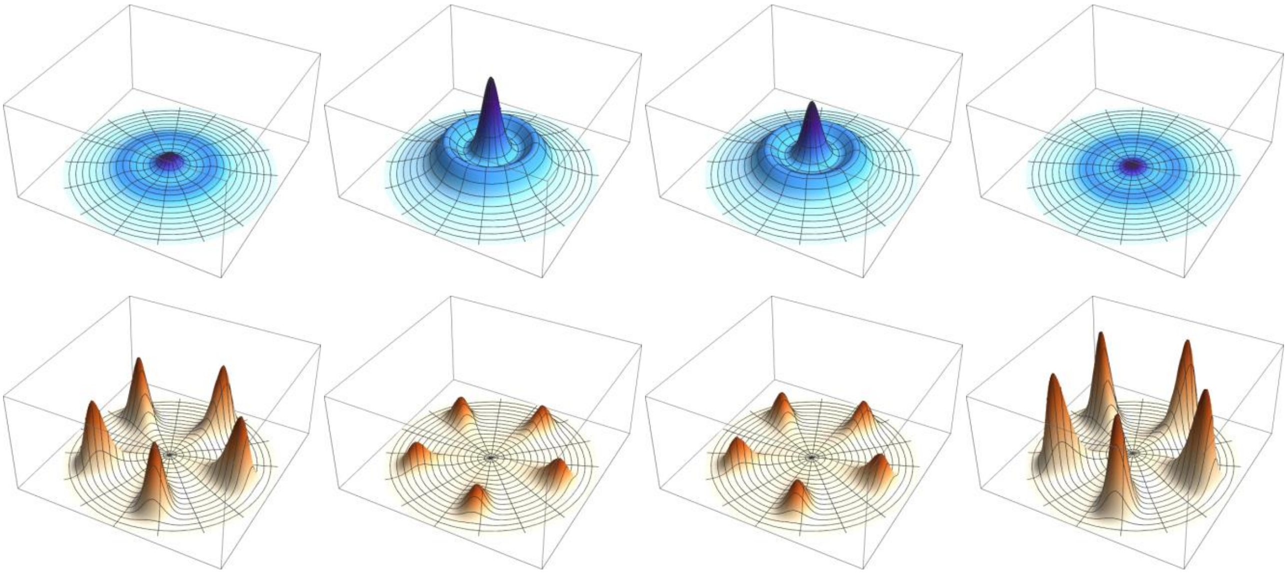


Source: Author's representation

The purpose of this example is to indicate that the adversarial system given by relations 16 and 38 can describe complex characteristics such as the domination of a population (species, class, society) over another and, also, the conflict between a centralized community and its periphery. In accordance with Figure 11, the dominant component of the system is Q^2, signified as orange (both lines and surfaces) and two scenarios presented, namely of Q^2 being peripherally (Figure 12) and centrally (Figure 13) concentrated. For Q^1, the opposite concentration is being applied.

Figure 12: Representation of the equation of distribution of Figure 11, where the component Q^2 is peripherally allocated and Q^1 and Q^2 correspond to the blue and orange surfaces respectively.

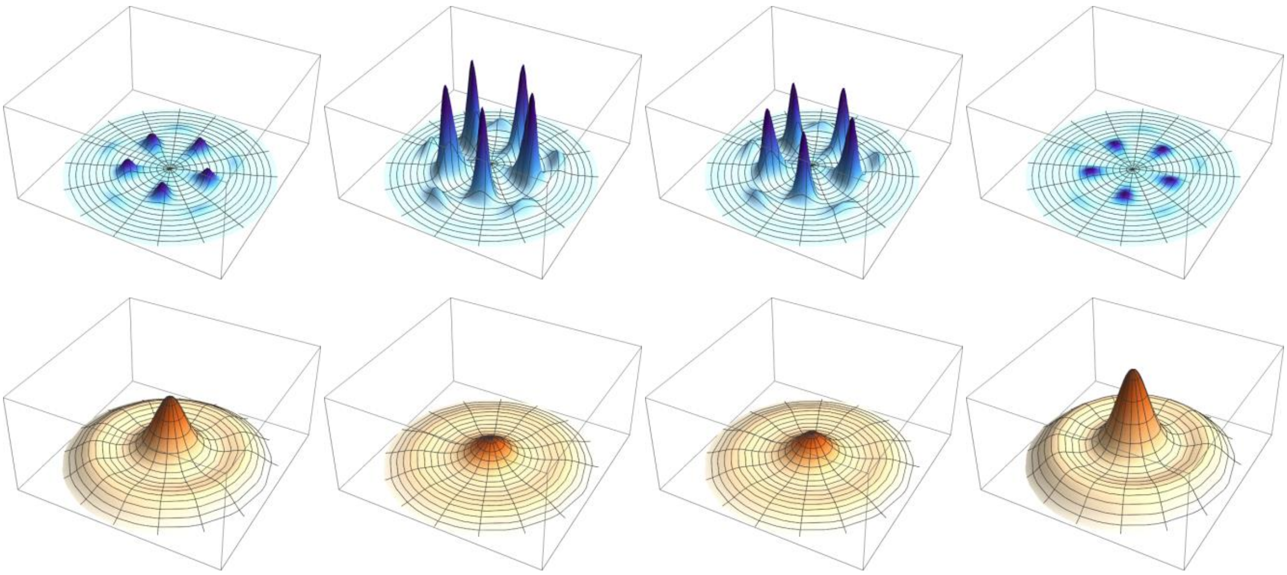
(a) t_0 (b) t_2 (c) t_3 and t_1 (d) t_4



Source: Author's representation

Figure 13: Representation of the equation of distribution of Figure 11, where the component Q^2 is centrally allocated and Q^1 and Q^2 correspond to the blue and orange surfaces respectively.

(a) t_0 (b) t_2 (c) t_3 and t_1 (d) t_4



Source: Author's representation

The information about the type of population concentration (central or peripheral) is introduced to relation 38 (and relation 40) via the Bessel functions $J_{n^i}(\rho\sqrt{K^i})$ and $Y_{n^i}(\rho\sqrt{K^i})$ and especially by the values given to the constants n^i. Indeed, the value $n^i = 0$ represents an isotropic distribution, corresponding to a mainly centrally allocated population and any value $n^i > 0$ produces a non-isotropic and mainly peripherally allocated population, thus, in Figure 12, $n^1 = 0$ and $n^2 = 5$, whereas in Figure 13, $n^1 = 5$ and $n^2 = 0$. For reasons of comparison, the corresponding component of the equation of distribution of the system remains the same for both Figures 12 and 13, except for the values of the constants n^i and, furthermore, the dimensions of the habitat (domain) are the same for all cases of both Figures. It can be noticed that the use of Bessel function to simulate the interaction between centrally and peripherally allocated populations can be applied to other systems, for example to the symbiotic systems of relation 34 or 37.

Two clarifications should be made concerning the interpretation of the results of the equations of motion and distribution, as shown in the Figures of this section:

- The reduction or elimination of any component of the system should not necessarily be conceived as the biological extermination of the corresponding population but, rather, to be interpreted as the partial or total absence of interaction between this population with the system (the other components) and, thus, being irrelevant.
- As previously mentioned, Figures 12 and 13, and thus, the corresponding relations 35 and 40 represent the simplest behavior pattern of relations 34 and 38 respectively. Indeed, the terms including Bessel functions are formed as products of one Bessel, one angular and one temporal function. Since Bessel and angular functions (as functions of coordinates) determine the magnitudes and positions of the local extremums of the distribution of each component, then the proper manipulation of the temporal functions can set these extremums in motion. An example of such a displacement is given in Figure 9 (angular motion) between Figures 9.d and 9.f for Q^1 (blue surface) or between Figures 9.b and 9.h for Q^2 (orange surface).

5. Discussion and conclusion

The effort to understand and try to deduce some logical patterns from the phenomenon of two or more interactive societies can be traced back to the works of Herodotus, Thucydides or Xenophon who were well aware of the importance of this understanding for the survival of the Greek civilization and, indeed, the major importance of the multiple dimensional Prey-Predator systems to Regional Science, Social and Economic, Political etc., interactions can be summarized by the Heraclitean maxim “War is the Father of all Things” (see for example Batty (2021), Batty (2022), Storper (2022) or Barthelemy (2023)). The mathematical formulation of such situations was given by Lotka (1925) and the deceptively simple simultaneous Lotka - Volterra equations (relation 23), initially applied to two-dimensional prey – predator biological systems, where all the components follow a periodic behavior, such as the data retrieved for Hadson Bay Trading Company, applied to the populations of Lynxes and Hares for a long time period. As shown by the references in section 3, this formulation nowadays also includes social, economic and political periodic systems as well as biological ones.

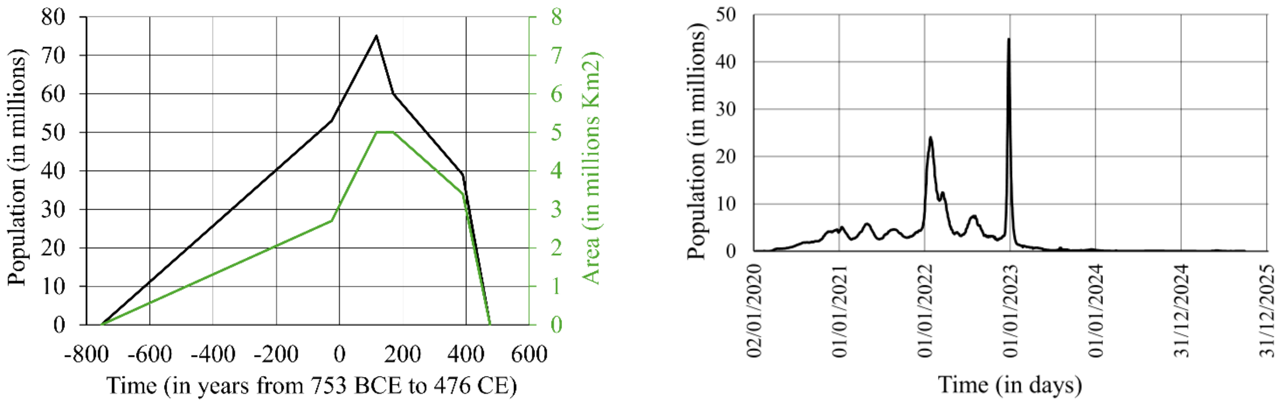
The herein prey – predator system described in sections 3 and 4 is not derived from the Lotka – Volterra equations are as application of the mathematical model described in Elias 2023 and 2024, a summary of which is given in section 1. In this way:

- an analytical solution can be derived,
- both the temporal evolution and the spatial expansion are investigated, in section 3 and 4 respectively (as already indicated in Louf and Barthelemy (2022), Rey and Anselin (2022) or Wei et al. (2023) and others),
- in this way the geographical aspects of some characteristics of Regional Science, such as multicentricity (see Elias (2023), Schmitt and Volgmann (2022)), anisotropic periphery development and divercification, (Partridge and Rickman (2022), Crescenzi and Iammarino (2023), Boschma (2023)) or self-organized cities (Portugali (1997)) can be studied and planned by the application of Calculus of Variations to the constitutional spatiotemporal equations of the herein presented model and
- the increase of the dimension of the system can immediately be achieved without proportional increase in the complexity of the constitutional equations.

As far as the equation of motion is concerned, apart from the purely periodic behavior (see Figures 2 and 5), another one emerges, that of a rise and fall of the system (see Figures 4, 6, 7, 11). In Figure 14, the latter behavior is depicted, where Figure 14.a and Figure 14.b correspond to Figure 7.a and Figure 6.a respectively, of the component Q^2 (orange line). In Figure 14.a, the simultaneous rise and fall of both population and area can be noticed, which can be compared to Figure 12 (Roman Empire corresponds to the blue surface).

Figure 14: Two examples of competitive systems a) from history and b) from epidemiology.

(a) Evolution of the Roman Empire (b) COVID-19 worldwide reported cases



Source: Author's representation (Data retrieved by a) ChatGPT, Wikipedia and b) World Health Organization)

There is a very interesting group of applications concerning economic characteristics of a society, such as:

- the partition of the population by their wealth or income, creating three or more distinct classes interacting with each other and to their environment,
- the distribution of the wealth of a society to public, personal and private,
- the interrelation between the wealth (or GNP) of nations and of larger international formations, such as European Union, Africa or BRICS.

These and other characteristics can be considered as components of the model demonstrated in sections 3 and 4, since the dimensions of the system can be easily expanded to be more than 2 (see section 2), to include multiple preys and predators. Moreover, all the necessary statistical data, with sufficient time depth (no less than 30 years period) and are ready for use, on the web site World Inequality Database. Nevertheless, not one such example is included in this paper, since "correlation does not imply causation", that is to say that some data tables, the application of Numerical Analysis and the presentation of few graphs does not constitute sufficient economic or social analysis. If the herein presented model could have indeed some use, this should be applied to various disciplines and real-case situations, along with a qualitative scrutinization of the calculation results and feedback, with the collaboration with Economists and/or Sociologists.

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