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A MATHEMATICAL MODEL FOR POPULATION DISTRIBUTION II: LINEAR SYSTEMS

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Biographical Note

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Abstract

The present paper constitutes a continuation of a previous relevant paper, entitled “A mathematical model for population distribution” by Elias (2023), in which the general theoretical model was described, and some initial applications were presented, namely some approximations of population distribution of a one-dimensional inertial system and a special case of one-dimensional dynamic system. Herein, by using the above model, the following issues will be addressed: a) the examination of multidimensional linear systems, both inertial and dynamic, facilitating the study of polycentric cities and systems of multiple cities (metropolitan areas) and b) the variation of the population distribution due to the geographical diversifications of the habitat of the system, as in a sloped terrain or of coastal cities. For each of the above cases, their behaviour is presented by producing and analysing the corresponding equations of motion and distribution and, whenever possible, an effort has been made to qualitatively compare the theoretical results to field data.

Keywords: population distribution, polycentric cities, geographical diversification
JEL Classification: Y80

1. Introduction

The behavior of the population of a city or a collection of cities, that is, its evolution over time and its expansion in the geographic space, determines the social behavior of this city and, at the same time, is determined by it. The economic, functional and cultural characteristics of the city can be traced and understood by the knowledge of how the density of its population vary in relation to time, space and the variation of the environment. Crucial functions, such as the transportational network, the operational zoning (including housing, commerce, public areas etc.) or vital programs referred to education, security and general quality of life, can be optimized of collapse depending on the profound understanding of the behavior of the density of the population, or the lack of it (e.g. Viguie (2017), Ayala, Martin-Roman and Vincente (2019), Castells-Quintana, Royuela and Veneri (2019), Abozeid and AboElatta (2021), Dentinho, Kourtit and Nijkamp (2021) or DiBartolomeo and Turnbull (2021)).

The necessity of improve the understanding such functions led to the creation of comprehensive mappings of the density of the population in various existing cities around the world (e.g. Bertaud and Malpezzi (2003), Bergmann (2019), Subasinghe, Wang and Murayama (2022) or Smith (2023)) and the derivation of analytical and statistical models and formulations (e.g. Clark (1951), Zielinski (1980), Griffith and Wong (2007), You (2017), Lang, Long and Chen (2018), Volpati and Barthelemy (2020), Feng and Chen (2021), Subasinghe, Wang, and Murayama (2022)) to corelate the density of the population of a city, or a collection of cities, to time and geographical space. One analytical model is described in Elias (2023), attempting to provide some general constitutional equations studying the spaciotemporal behavior of the population distribution, by using the smallest possible number of axiomatic principles. In the present paper, some applications of this model are presented, namely the investigation of inertial and dynamic linear systems and the influence of topographical irregularities on population distribution.

An attempt to understand the behavior of even the "simpler" society requires the determination and analysis of a multitude of interacting factors, so that the derivation of a formalization "from within" seems futile, at least from the point of view of an engineer. The formalization "from above" of some social characteristics (an "in vitro society") can be achieved, by the axiomatic acceptance that a subset $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$ of the above set of the interacting factors exists, so that the members of $\boldsymbol{Q}$ are linearly independent to each and, also, that $\boldsymbol{Q}$ adequately approximates this society. This axiom permits one to establish a space of reference for the social system, which has dimension N, a base (the natural base): $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$ and a Riemannian metric (in general): $(ds)^2 = g_{ij} dQ^i dQ^j$ $(i, j = 1, \dots, N)$, where the functions $g_{ij} = g_{ij}(Q^1, \dots, Q^N)$ are the components of the metric tensor.

By this device, any social event of the system can be represented by a point of its space of reference, so that the history of the system coincides to a curve embedded into the space of reference, namely the trajectory $\boldsymbol{Q}(s) = \left(Q^1(s), \dots, Q^N(s)\right)$, where the parameter s is the arc length of the curve. The application of the Principle of the Least Action produces the exact form of the trajectory, as a solution of the simultaneous differential equations:

$$\frac{d^2Q^i(s)}{(ds)^2} + \Gamma^i_{jk}\frac{dQ^j(s)}{ds}\frac{dQ^k(s)}{ds} = 0 \tag{1}$$

which is the geodesics of the space of reference, where:

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mj}}{\partial Q^k} + \frac{\partial g_{mk}}{\partial Q^j} - \frac{\partial g_{ij}}{\partial Q^m}\right) \tag{2}$$

is the Christoffel symbol of the second kind.

It should be noticed that the temporal coordinate $x^0 \equiv t$ or the spatiotemporal coordinates $\boldsymbol{x} = \{x^0, x^1, x^2, x^3\}$ which the usual base of a geographical four-dimensional Euclidean space, are not included in any base $\boldsymbol{Q} = \{Q^1, \ldots, Q^N\}$ of the space of reference, since the behavior of a social system is independent of the values of both time or geographical space. Indeed, let us consider a completely empty, homogeneous and isotropic field, then the placement of the same society at different geographical coordinates and time would not produce any variation in the behavior of this society. Any external influence, in the form of geographical diversifications or even other social or ecological systems are not inner characteristics of the first society but forces acting upon it, hence, these influences should be treated as boundary conditions applied on its constitutional differential equations. In section 3, two examples of incorporating geographical diversification into the constitutional equations are presented.

The transformation between the arc length s and the time t in the trajectory deduces the equation of motion of the system:

$$\frac{d^2Q^i(t)}{(dt)^2} + \Gamma^i_{jk}\frac{dQ^j(t)}{dt}\frac{dQ^k(t)}{dt} = 0 \tag{3}$$

describing its temporal behavior. The spatiotemporal character of the system in given by the densities $Q^i(\boldsymbol{x})$ of the components of the base, defined as:

$$Q^i(x^0) = \int_{\Phi} Q^i(\boldsymbol{x})d\Phi : d\Phi = dx^1dx^2dx^3 \tag{4}$$

so that the equation of distribution is derived as follows:

$$\sum_{\mu=0}^{3}\left(\frac{\partial^2 Q^i(\boldsymbol{x})}{\partial x^\mu \partial x^\mu} + \Gamma^i_{jk}\frac{\partial Q^j(\boldsymbol{x})}{\partial x^\mu}\frac{\partial Q^k(\boldsymbol{x})}{\partial x^\mu}\right) = 0 \Rightarrow \Delta Q^i(\boldsymbol{x}) + \sum_{\mu=0}^{3}\left(\Gamma^i_{jk}\frac{\partial Q^j(\boldsymbol{x})}{\partial x^\mu}\frac{\partial Q^k(\boldsymbol{x})}{\partial x^\mu}\right) = 0 \tag{5}$$

where the Laplace operator of relation 5 is, in general, four-dimensional (spaciotemporal).

The last remaining element for the constitutional equations of relations 3 and 5 to be fully determined is the exact form the metric tensor, which can be deduced by a third axiom, namely the Principle of Equivalence. This axiom states that the external force acting upon a system is equivalent to the curvature of the space of reference of this system, so that in the case of an inertial (no force) system, the space of reference is Euclidean, that is, a base $\boldsymbol{U} = \{U^1, \ldots, U^N\}$ of the space (the usual base) exists, such that, the metric form becomes: $(ds)^2 = \delta_{ij} dU^i dU^j = dU^i dU^i$ $(i, j = 1, \dots, N)$. Hence, the constitutional equations of an inertial system, as described by its usual base, takes the forms of:

$$\frac{d^2U^i(t)}{(dt)^2} = 0 \; and \; \Delta U^i(\boldsymbol{x}) = 0 \tag{6}$$

therefore, the force $\boldsymbol{F}(t) = \left(F^1(t), \dots, F^N(t)\right)$ and the stress $\boldsymbol{F}(\boldsymbol{x}) = \left(F^1(\boldsymbol{x}), \dots, F^N(\boldsymbol{x})\right)$ of a dynamic system, when described by the usual base of the system, are given by:

$$F^i(t) = -\Gamma^i_{jk} \frac{dU^j(t)}{dt} \frac{dU^k(t)}{dt} \; and \; F^i(\boldsymbol{x}) = -\sum_{\mu=0}^{3} \left(\Gamma^i_{jk} \frac{\partial U^j(\boldsymbol{x})}{\partial x^\mu} \frac{\partial U^k(\boldsymbol{x})}{\partial x^\mu} \right) \tag{7}$$

An analytical treatment of the physical and mathematical aspects of the above conclusions can be found in i.e. Gelfand and Fomin (1963), Weinstock (1974), Landau and Lifshitz (1980), Dodson and Poston (1997), Francoise, Nabel and Tsun (Editors) (2006), Itskov (2007), Talman (2007) or Bourles (2019).

The population system is the limited case of a social system, where each component of the natural base of its space reference represents populations, that is, quantities obeying the Kolmogorov simultaneous equations (i.e. Smirnov (1964)):

$$\frac{dQ^i(t)}{dt} = f^i\left(Q^1(t), \dots, Q^1(t)\right) : i = 1, \dots, N \tag{8}$$

In the restricted situation in which the population system is inertial and one-dimensional, relation 8 reduces to the Malthus equation:

$$\frac{dQ(t)}{dt} = CQ(t) \Rightarrow Q(t) = exp(At + B) \tag{9}$$

where A, B and C are a real constant. The comparison between relations 6 and 9 deduces the acceptable transformation between the natural and the usual base of a population system:

$$Q(U) = exp(U) \tag{10}$$

It can be noticed that relation 10 denotes an admissible transformation (bijective function) for all values of U and for all positive values of Q, since there can be no negative values for a population.

The simultaneous manipulation of two bases, namely the usual and the natural, is essential for the complete understanding of a system. Indeed, described by the usual base, the constitutional equations take their simplest form and the external and internal influences are expressed in their purest expression, without terms derived from base transformation. On the other hand, the components of the usual base do not include the physical attributes and characteristics of the system, which only the natural base can achieve. For example, in the case of an inertial population component, its description by the usual base produces an equation of motion which is linear with respect to time and although this equation is the simplest one, cannot produce the main characteristic of the population, that is its exponential growth over time. This characteristic is given when the same component is referred to its natural base, produced by the transformation of relation 10. Evidently, another transformation of the usual base can correspond to some base components having entirely different characteristics (economic, demographic etc.).

2. Definition of linear systems with constant coefficients

Let the starting point be an N-dimensional inertial population system, where no component of its base interacts with any other component or with the environment of the system. In this case, a base of its space reference exists, namely the usual base $\widetilde{\boldsymbol{U}} = \{\widetilde{U}^1, \dots, \widetilde{U}^N\}$, for which the metric form becomes Pythagorean: $(ds)^2 = d\widetilde{U}^i d\widetilde{U}^i$ $(i = 1, \dots, N)$, as mentioned in the previous section. The existence of any influence to the components of the base $\widetilde{\boldsymbol{U}}$ can be introduced by a function of the form $\widetilde{U}^{N+1} = \widetilde{U}^{N+1}(\widetilde{U}^1, \dots, \widetilde{U}^N)$, where the component $\widetilde{U}^{N+1}$ is a dummy one (temporarily used to facilitate the calculations). Hence, the metric form incorporates the influence as follows:

$$\begin{aligned}(ds)^2 = d\widetilde{U}^{N+1} d\widetilde{U}^{N+1} + \sum_{i=1}^{N} \left(d\widetilde{U}^i d\widetilde{U}^i\right) = \sum_{i,j=1}^{N} \left(\left(\delta_{ij} + \frac{\partial \widetilde{U}^{N+1}}{\partial \widetilde{U}^i} \frac{\partial \widetilde{U}^{N+1}}{\partial \widetilde{U}^j}\right) d\widetilde{U}^i d\widetilde{U}^j\right) \\ = g_{ij}\left(\widetilde{U}^1, \dots, \widetilde{U}^N\right) d\widetilde{U}^i d\widetilde{U}^j : i, j = 1, \dots, N\end{aligned} \tag{11}$$

where, in general, the metric is non-Euclidean and the metric tensor is a function of the components of the base.

In the case that the influence can be approximated by a linear form, that is when $\widetilde{U}^{N+1} = \widetilde{U}^{N+1}\left(\widetilde{U}^1, \dots, \widetilde{U}^N\right)$ is a linear function of the components of the base, all the components of the metric tensor are real constants. Then, a base transformation of the form $\widetilde{U}^i = \widetilde{U}^i\left(\widetilde{\widetilde{U}}^1, \dots, \widetilde{\widetilde{U}}^N\right)$ there always exists, such that the metric form of relation 11 to be reduced to a diagonal metric form: $(ds)^2 = \xi_i d\widetilde{\widetilde{U}}^i d\widetilde{\widetilde{U}}^i$, where the quantities ξ_i are real but non necessary positive constants. A further of reduction of the metric form to the Pythagorean metric: $(ds)^2 = dU^i dU^i$ can be achieved by the application of a second transformation $dU^i = \lambda^i d\widetilde{\widetilde{U}}^i$ (all $d\widetilde{\widetilde{U}}^i$ being unit vectors), where the eigenvalues $\lambda^i \equiv \sqrt{\xi^i}$ can have real or complex constant values It should be noted that since the system and, consequently, its space of reference is N-dimensional, there exist exactly N distinct and non-vanishing eigenvalues. For the arc length ds of the trajectory to take only real values, all complex quantities λ^i should be in pairs of complex conjugates. This last base $U = \{U^1, \dots, U^N\}$ can be considered the usual base of the system, which incorporates both the external and internal influences expressed by relation 11. The existence of complex eigenvalues "informs" the system of its dynamic character, since they alter the space of reference from Euclidean to Pseudo-Euclidean. A detailed treatise on this subject and on Hermitian forms can be found in i.e. Sokolnikoff (1964), Lang (1995), Dodson and Poston (1997).

It can be noticed that in either cases of Euclidean or Pseudo-Euclidean space, the components of the metric tensor are constants, either real of complex, hence all the components of the Christoffel symbols vanish, reducing the equations of motion and distribution for the base $\boldsymbol{U} = \{U^1, \dots, U^N\}$ as in relation 6, that is: $U^i(t) = \lambda^i t + c^i$ and $\Delta U^i(\boldsymbol{x}) = 0$, where c^i are real constants. Since the metric form when described by the usual base is Pythagorean, the transformation of relation 10 can be applied, given the simplest natural base of a population system: $\boldsymbol{P} = \{P^1, \dots, P^N\}$, where $P^i = exp(U^i)$ and the components of this base can be used as the building blocks for more complicated population system. The constitutional equations of an inertial system, where all the eigenvalues λ^i are real constants, are given by:

$$P^i(t) = exp\left(U^i(t)\right) = C^i exp(\lambda^i t) \Rightarrow \frac{dP^i(t)}{dt} = \lambda^i P^i(t) \tag{12}$$

for the equation of motion, where $C^i = exp(c^i)$ are real constants and

$$P^i(\boldsymbol{x}) = exp\left(U^i(\boldsymbol{x})\right) \colon \Delta U^i(\boldsymbol{x}) \tag{13}$$

for the equation of distribution.

For the dynamic system, some complex eigenvalues exist, forming pair of complex conjugates, such that: $\lambda^i = \alpha^i + I\beta^i$, where $I = \sqrt{-1}$ represent the imaginary unit, for clarity reasons. Hence, $U^i(t) = \alpha^i t + I\beta^i t + c^i$ and the equation of motion for the base $\boldsymbol{P}$ becomes:

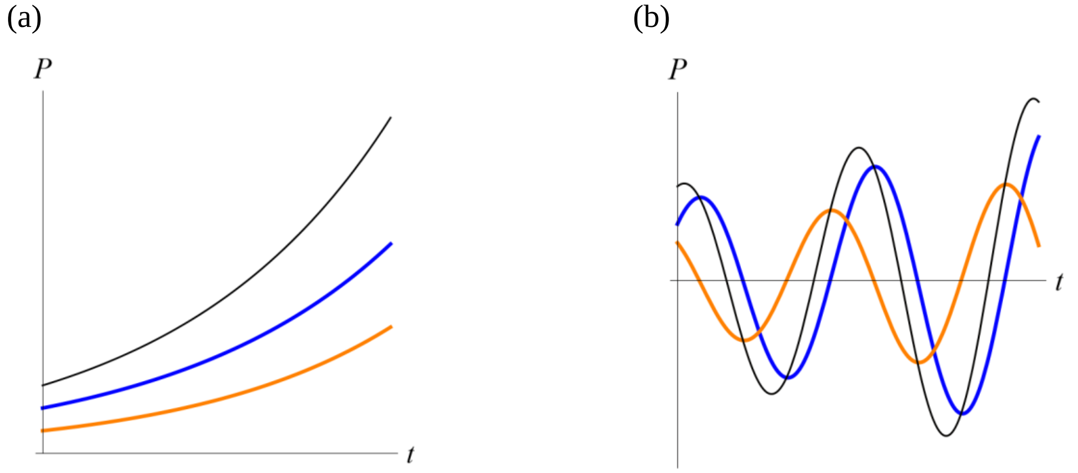
$$P^i(t) = C^i exp(\lambda^i t) = C^i exp(\alpha^i t)\left(cos(\beta^i t) + Isin(\beta^i t)\right) \tag{14}$$

where α^i, β^i and c^i are real constants. In an analogous manner, the equation of distribution has been informed for the dynamic state of the system by being constructed as a complex function: $U^i(\boldsymbol{x}) = U^i_{RE}(\boldsymbol{x}) + IU^i_{IM}(\boldsymbol{x})$, where $U^i_{RE}(\boldsymbol{x})$ and $U^i_{IM}(\boldsymbol{x})$ are the real and the imaginary parts respectively. The existence of a complex function $U^i(\boldsymbol{x})$ indicates that exactly one other such function exists that is the complex conjugate of the former. The equation of distribution for the base $\boldsymbol{P}$ is given by:

$$\begin{gathered} P^i(\boldsymbol{x}) = exp\left(U^i(\boldsymbol{x})\right) = exp\left(U^i_{RE}(\boldsymbol{x})\right)\left(cos\left(U^i_{IM}(\boldsymbol{x})\right) + Isin\left(U^i_{IM}(\boldsymbol{x})\right)\right) \\ where\ \Delta U^i(\boldsymbol{x}) = 0 \Rightarrow \Delta U^i_{RE}(\boldsymbol{x}) = \Delta U^i_{IM}(\boldsymbol{x}) = 0 \end{gathered} \tag{15}$$

Relations 14 and 15 are derived by applying Euler's formula to relation 12 and 13 respectively. It can be noticed that the components $P^i(t)$ of the dynamic system (Figure 1(b)) vanish after a short period of time, leading the system to collapse. This inconsistency is nullified when the system takes its general form.

Figure 1. Representation of two components $P^1(t)$ and $P^2(t)$ of the equation of motion and the total population $P(t) = P^1(t) + P^2(t)$, with blue, orange and black solid lines respectively. The inertial case of relation 12 is shown in Figure 1(a) and the dynamic one of relation 14 in Figure 1(b).



Source: Author's representation

The most general N-dimensional linear population system with constant coefficients has the following form:

$$\frac{dQ^i(t)}{dt} = A^i_j Q^j(t) \Rightarrow \begin{bmatrix} dQ^1(t)/dt \\ \vdots \\ dQ^1(t)/dt \end{bmatrix} = \begin{bmatrix} A^1_1 & \cdots & A^1_N \\ \vdots & \ddots & \vdots \\ A^N_1 & \dots & A^N_N \end{bmatrix} \begin{bmatrix} Q^1(t) \\ \vdots \\ Q^N(t) \end{bmatrix} \tag{16}$$

where all A^i_j are real constants. Both relations 12 and 16 are special cases of Kolmogorov equations in relation 8 and describe the same system (that defined by the metric form in relation 11) using different natural bases. Indeed, by starting the base $\boldsymbol{P}$, a linear transformation: $Q^i = B^i_j P^j$ can be applied, where B^i_j are real constants, which leads to a now, more general, natural base $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$. The matrix $\boldsymbol{A} = [A^i_j]$ of relation 16 is derived by the matrix transformations $\boldsymbol{B} = [B^i_j]$ and $\boldsymbol{\Lambda} = [\lambda^i]$ (diagonal matrix) as follows: $\boldsymbol{A} = \boldsymbol{B\Lambda B}^{-1}$, that is: $A^i_j B^j_k = B^i_k \lambda^k$. Finally, when referred to the natural base $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$, the equation of motion is given by:

$$Q^i(t) = B^i_k P(t) = B^i_k C^k exp(\lambda^k t) \tag{17}$$

and the equation of distribution, by:

$$Q^i(\boldsymbol{x}) = B^i_k P^k(\boldsymbol{x}) = B^i_k exp\big(U^k(\boldsymbol{x})\big) : \ \Delta U^k(\boldsymbol{x}) = 0 \tag{18}$$

where the constants B^i_j and C^i are real and each eigenvalue λ^i can be either real or complex (in the latter case the Euler's formula is applied).

The form of relation 18 permits the problem of the distribution of one or many populations to be reduced to the derivation of a proper particular solution of the Laplace equation: $\Delta U^k(\boldsymbol{x}) = 0$, expressed to a three-dimensional geographic space, including the time and two geographic coordinates: $\boldsymbol{x} = \{x^0, x^1, x^2\}$. In this and the next sections the habitat of the population is assumed to be a flat (regular) two-dimensional surface. The introductions of geographical irregularities will be addressed in section 4, where the Laplace equation will be replaced by Laplace Beltrami equation. There who main categories by which a city population can be distributed, the first of which is the (nearly) homogeneous development, where the general city plan is mostly orthonormal (Hippodamian) in which case the geographic base use Cartesian coordinates: $\boldsymbol{x} = \{t, x, y\}$, leading to the following expression of Laplace equation:

$$\Delta U^i(t,x,y) = \frac{\partial^2 U^i(t,x,y)}{\partial t \partial t} + \frac{\partial^2 U^i(t,x,y)}{\partial x \partial x} + \frac{\partial^2 U^i(t,x,y)}{\partial y \partial y} = 0 \qquad (19)$$

The second category is that of (nearly) isotropic development where the general city plan is mostly radial (centralized). This category is better described by a polar coordinate system, having a base: $\boldsymbol{x} = \{t, \rho, \theta\}$ where ρ is the polar radius from the hypothetical center of the city and θ is the azimuthal angle. The Laplace equation in this case takes the expression:

$$\Delta U^i(t,\rho,\theta) = \frac{\partial^2 U^i(t,\rho,\theta)}{\partial t \partial t} + \frac{\partial^2 U^i(t,\rho,\theta)}{\partial \rho \partial \rho} + \frac{1}{(\rho)^2}\frac{\partial^2 U^i(t,\rho,\theta)}{\partial \theta \partial \theta} + \frac{1}{\rho}\frac{\partial U^i(t,\rho,\theta)}{\partial \rho} = 0 \qquad (20)$$

It is obvious that most cities, especially the historical ones, are the product of the incorporation of areas of both categories (ancient and modern parts of the city) or of an amalgamation. An analytical study and the derivation of complete solutions of Laplace equation can be found in i.e. Smirnov (1964), Pinchover and Rubinstein (2005), Taylor (2011) or Sauvigny (2012).

3. Analysis of linear systems with constant coefficients

The formulation of the previous section permits the production of any linear system as a linear combination of the building blocks $P^i(\boldsymbol{x})$, thus reducing the problem of the population distribution to the solution of the Laplace equations in relation 19 or 20. In the case of a inertial system, the boundary conditions of the Laplace equation, derived by the observations and the models referred in section 1, are as follows:

- As mentioned in the previous section, the necessary and sufficient condition of a linear system to be inertial is for all the eigenvalues λ^i to take real values, which leads all the components of the equation of motion to be uniformly monotonous functions of time, as in relation 12 and Figure 1.a. Hence, due to the properties of the transformations in relation 13 and 18, any particular solution of Laplace equation of relations 19 or 20 should include only temporal terms that are also uniformly monotonous functions.

- The observations and the models mentioned at the references of section 1, indicate that the spatial terms of any particular solution should have a general tendency (pending some localized fluctuations) of decreasing population density as the distance from the hypothetical city center increases. Moreover, in reference to relation 20, the population density cannot depend on the polar angle θ in a monotonous manner but, rather, the terms include θ should be constant or periodic functions.
- In this section, the habitat of the system is supposed to be flat, so that the density is unaffected by any geographical features. The influence of population distribution due to geographical variations are dealt with in section 4.

An inertial component $P^i(\boldsymbol{x})$ of the equation of distribution, derived as a particular solution of relation 19, including the above boundary conditions, is as follows:

$$P^i(t,x,y) = exp\left(U^i(t,x,y)\right)$$

$$= exp\left(\left(C_1^i exp\left(t\sqrt{K_1^i}\right) + C_2^i exp\left(-t\sqrt{K_1^i}\right)\right)\left(C_3^i cos\left(x\sqrt{|K_2^i|}\right)\right.\right.$$

$$\left. + C_4^i sin\left(x\sqrt{|K_2^i|}\right)\right)\left(C_5^i cos\left(y\sqrt{|K_3^i|}\right) + C_6^i sin\left(y\sqrt{|K_3^i|}\right)\right) + \frac{K_4^i}{2}(t)^2 \qquad (21)$$

$$\left. + \frac{K_5^i}{2}(x)^2 + \frac{K_6^i}{2}(y)^2 + C_7^i t + C_8^i x + C_9^i y + C_{10}^i\right)$$

Such that $K_1^i + K_2^i + K_3^i = 0$ and $K_4^i + K_5^i + K_6^i = 0$

where $K_1^i, K_4^i \geq 0$ (according to the first boundary condition), and C_m^i are real constants. Since, according to the third boundary condition (flat habitat), the behavior of the population distribution has no reason to be altered according to the choice of the geographical coordinates x and y, the constants K_m^i become:

$$K_2^i = K_3^i = -K_1^i/2 \text{ and } K_5^i = K_6^i = -K_4^i/2 \qquad (22)$$

It can be noticed that the Laplace equation (relations 19 and 20) is a linear partial differential equation, therefore any linear combination of any particular solutions, such that is given in relation 21, is also a particular solution of the Laplace equation:

$$U^i(t,x,y) = A_1^i U^1(t,x,y) + A_2^i U^2(t,x,y) + \cdots \qquad (23)$$

where A_m^i are real constants. Due to relation 23, the equation of distribution can take a rather extensive form, which can be proven unnecessary, since the more manageable form of relation 21 can sufficiently describe the important characteristics of a city.

The purely isotropic behavior of the population distribution is given by a reduction of relation 21, such that all the constants from C_1^i to C_6^i vanish, so that only the exponent of a polynomial remains, as represented in Figure 2:

$$P^i(t,x,y) = exp\left(U^i(t,x,y)\right)$$
$$= exp\left(\frac{K_4^i}{2}(t)^2 - \frac{K_4^i}{4}(x)^2 - \frac{K_4^i}{4}(y)^2 + C_7^i t + C_8^i x + C_9^i y + C_{10}^i\right) \quad (24)$$

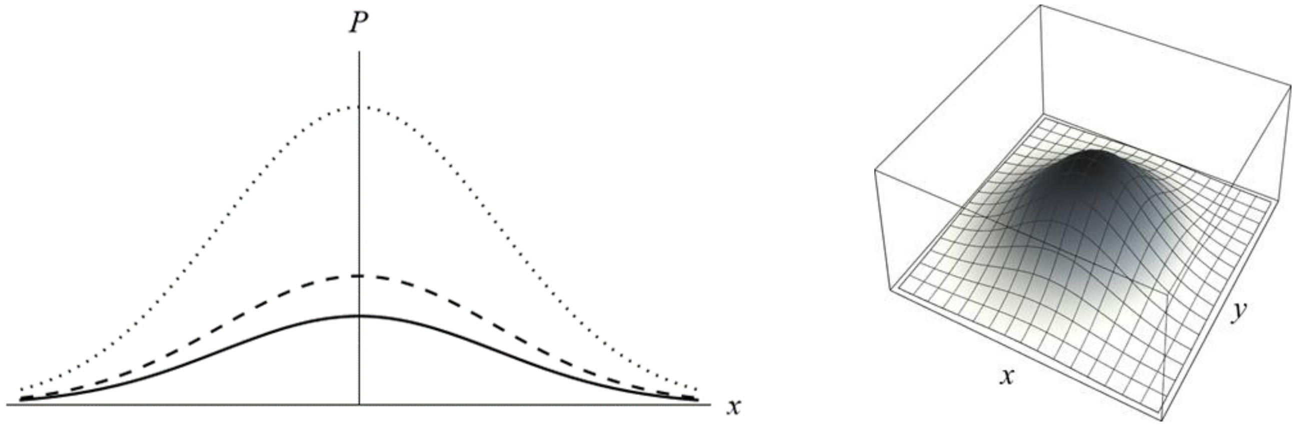
This last relation was mentioned, along with its derivation from the general model, in Elias (2023) and is a variation of the equations proposed by Clark (1951), Zienlinski (1980), Anselyn and Can (1986), Martori and Surinach (2001) or Griffith and Wang (2007), which take the general form:

$$P(\rho) = exp\left(\sum_{m=0}^{M}(C_m(\rho)^m)\right) \quad (25)$$

where the population distribution is considered as independent of time (time is constant) and ρ represents the polar radius from the hypothetical city center. It can be noticed that although the total population increases exponentially over time, this increase is not geographically homogeneous but, rather, tends to concentrate to the denser areas near the hypothetical center.

Figure 2. Representation of a partial application of relation 21, that is a purely isotropic distribution of a one-dimensional inertial system $P(\boldsymbol{x})$. a) Depiction of the population density along the x axis for three instants of time $t_1 < t_2 < t_3$ (solid, dashed and dotted lines respectively). b) 3D depiction of Figure 2-a, for a random instant, where darker areas correspond to higher density.

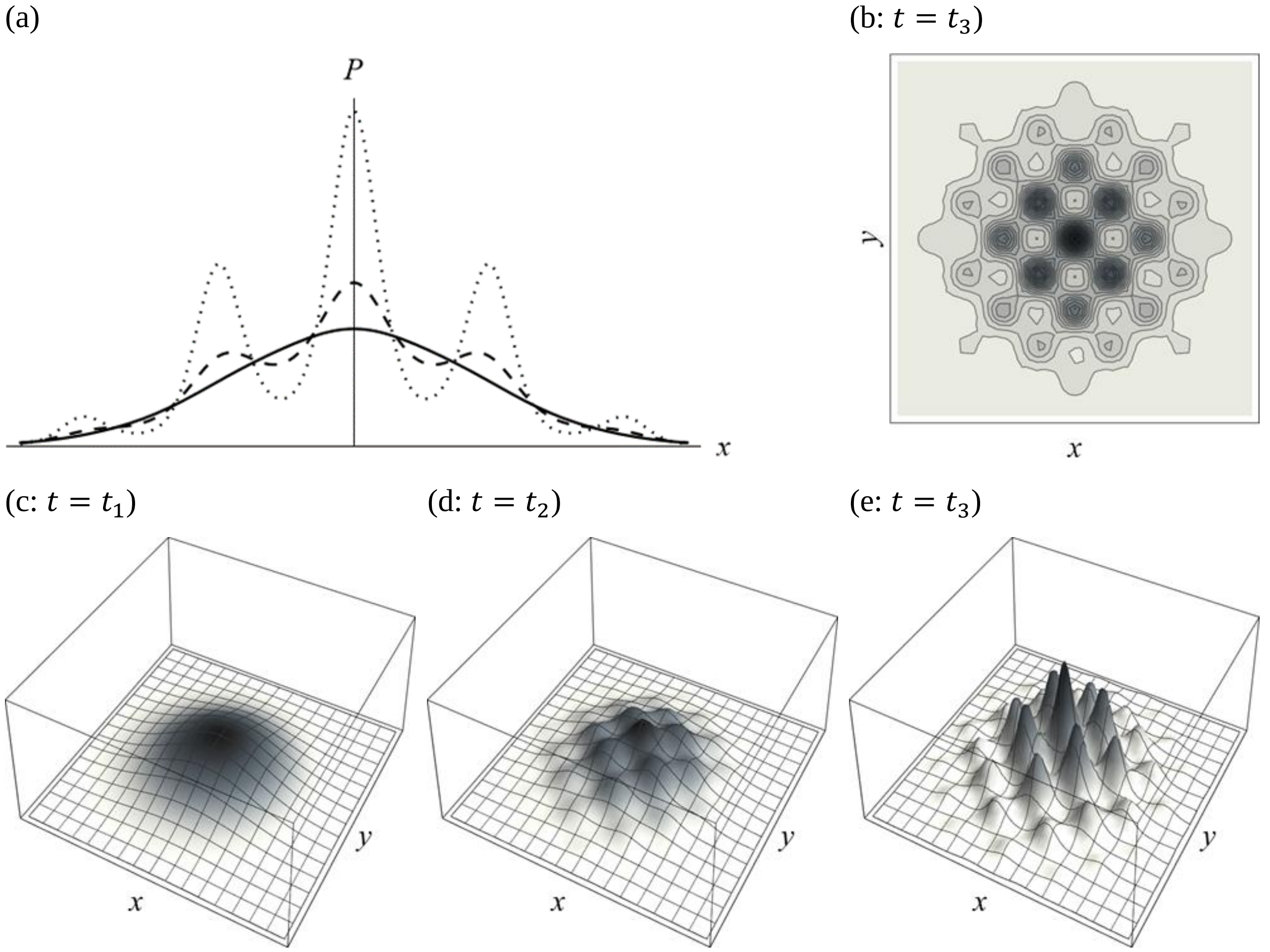
(a) (b)



Source: Author's representation

The complete application of relation 21 produces an unisotropic and unhomogeneous behavior and, pending the choices of the real constants C_3^i, C_4^i, C_5^i and C_6^i, lead to symmetrical (as in the following Figures 3 and 4) or asymmetrical population distributions, relative to the hypothetical city center, where $x = y = 0$.

Figure 3. Representation of the complete application of relation 21 where darker areas correspond to higher density. a) Depiction of the population density along the x axis for three instants of time $t_1 < t_2 < t_3$ (solid, dashed and dotted lines respectively). b) contour plot of the population density at the advanced state of the city ($t = t_3$). c, d, e) 3D depiction of Figure 3-a, for the initial, intermediate and advance state of the city respectfully.



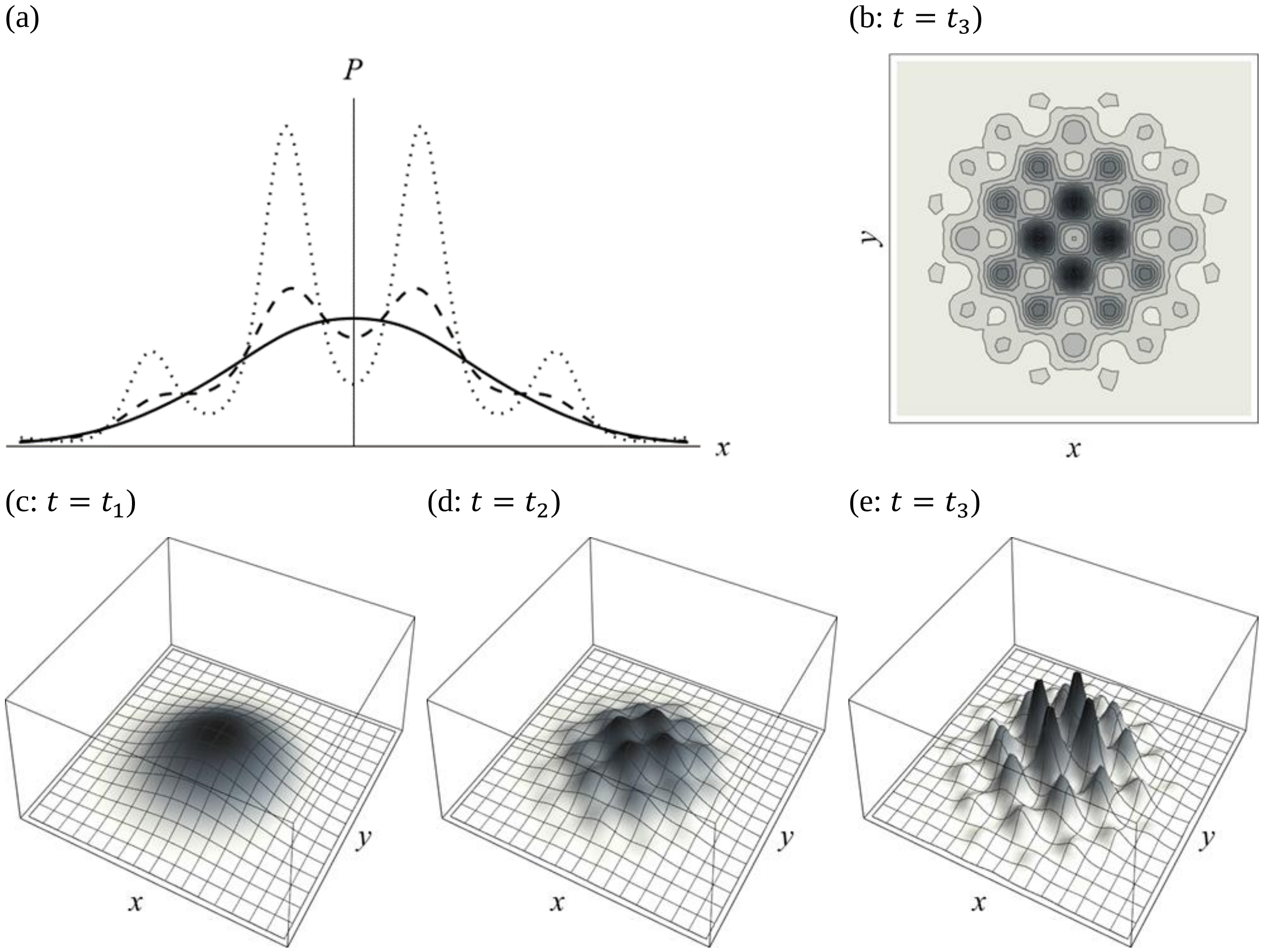
Source: Author's representation

The most important characteristic of relation 21 is that it can generate polycentric cities, since:

- Although the total population of the city increases exponentially, this increase is not homogeneous.
- In the initial state of the city (small values of time and population), the population density follows a uniformly monotonous decreasing function of the polar radius, as in the case of Figures 2(a), 3(c) and 4(c).
- In the intermediate state, the city tends to form neighborhoods, that is regularly arranged areas of local extremums of the population density, where one or more absolute maximum values occur at, or near, the city center, as in Figures 3(d) and 4(d).

- In the advanced state, the difference between the alternatively arranged local maximum and minimum values of the population density become more prominent, specifying separate neighborhoods, as in Figures 3(e) and 4(e).

Figure 4. Alternate representation of the complete application of relation 21 where darker areas correspond to higher density. a) Depiction of the population density along the x axis for three instants of time $t_1 < t_2 < t_3$ (solid, dashed and dotted lines respectively). b) contour plot of the population density at the advanced state of the city ($t = t_3$). c, d, e) 3D depiction of Figure 4(a), for the initial, intermediate and advance state of the city respectfully.



Source: Author's representation

It can be noticed that the only difference between Figures 3 and 4 is that, in the former there exists one major center (Figure 3(c)) and several minor (local) centers, as opposed to the latter, where exist several (in this case four) major centers replacing the initial center of Figure 4(c), and several local centers. The question as to whether the case of Figure 3 constitutes a polycentric city, in contrast to the well-defined polycentric city of Figure 4, is up to the decentralization (or not) of complex administrational, economic, cultural etc. networks (see, for example, Louf and Barthelemy (2013), Schmitt et al. (2015), Viguie (2017), Castell-Quintanna, Royuela and Veneri (2019), Abozeid and AboElatta (2021), Wu, Smith and Wang (2021), Derudder, Meijers, Harrison at el. (2022)). Obviously, the degree of interdependence between the geographical topology of these networks and the population distribution increases with respect to time (and consequently to population density). The most profound example is the optimization of the transportation network that, after the intermediate state of the city, can be distinguished in the contour plots (maps) of Figures 3(b) and 4(b), where the major city blocks are indicated by dark areas. An optimal transportation network is defined by the needs of the population (local density areas), in the intermediate state, but after some time (in the advanced state) it contributes to the formation of population distribution.

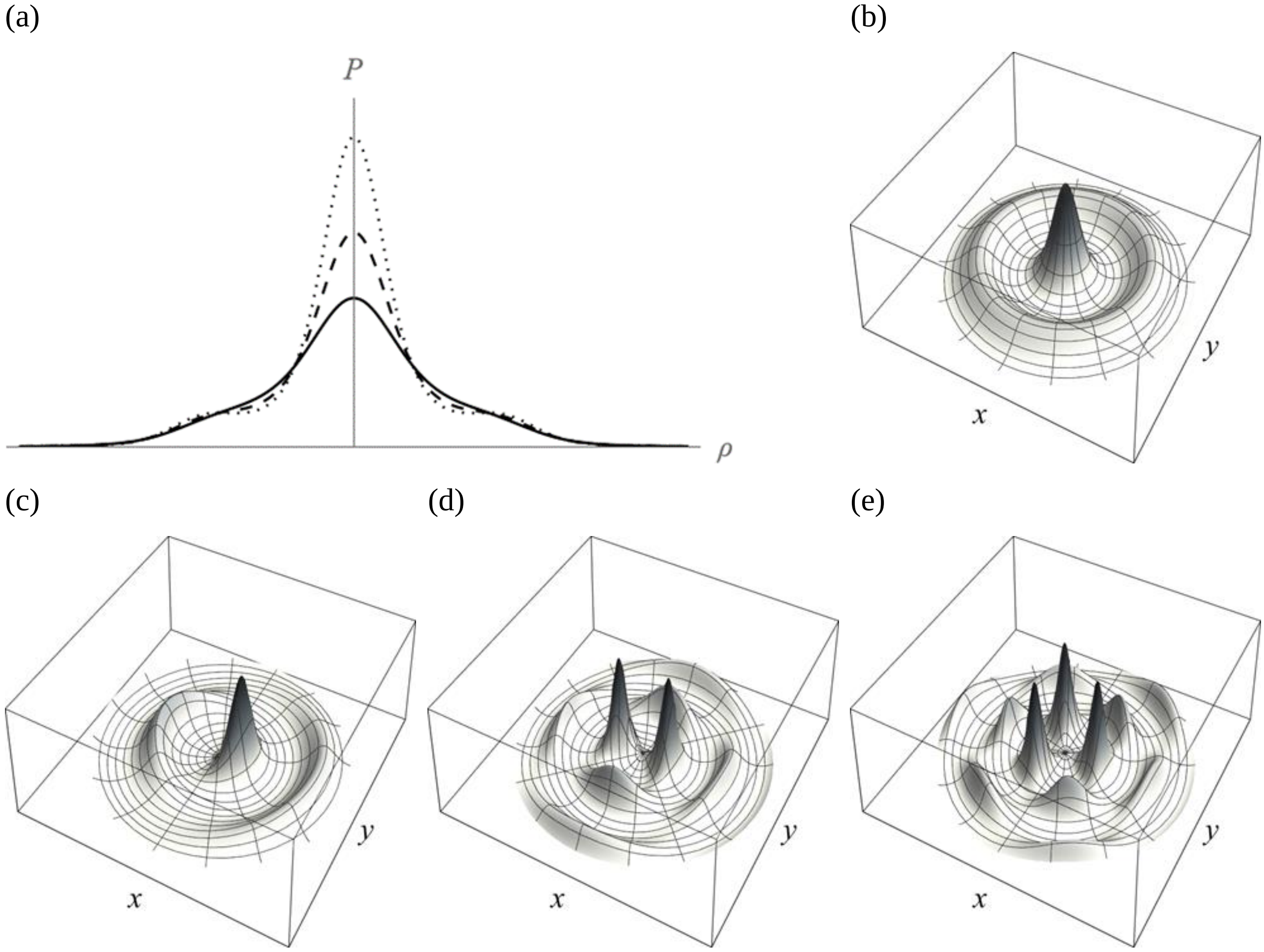
The choice of Cartesian coordinate system for the description of the geography of the habitat in the constitutional equations of relation 19 leads to equation of distribution in relation 21, which produce city blocks (urban plan) arranged in an orthonormal (Hippodamian) manner, but this arrangement is not unique. Indeed, by choosing the polar coordinate system, as in relation 20, the arrangement of the city blocks becomes centralized, expanding radially from the city center outward. By applying the same boundary conditions as in relation 19, the polar equation of distribution is given by:

$$\begin{aligned} P^i(t,\rho,\theta) &= exp\left(U^i(t,\rho,\theta)\right) \\ &= exp\Bigg(\left(C_1^i exp\left(t\sqrt{K_1^i}\right) + C_2^i exp\left(-t\sqrt{K_1^i}\right)\right)\Bigg(C_3^i J_{\sqrt{|K_2^i|}}\left(\rho\sqrt{K_1^i}\right) \\ &+ C_4^i Y_{\sqrt{|K_2^i|}}\left(\rho\sqrt{K_1^i}\right)\Bigg)\left(C_5^i cos\left(\theta\sqrt{|K_2^i|}\right) + C_6^i sin\left(\theta\sqrt{|K_2^i|}\right)\right) + \frac{K_3^i}{2}(t)^2 \\ &- \frac{K_3^i}{4}(\rho)^2 - \frac{K_4^i}{2}\left(ln(\rho)\right)^2 + \frac{K_4^i}{2}(\theta)^2 + C_7^i t + C_8^i ln(\rho) + C_9^i\theta + C_{10}^i\Bigg) \end{aligned} \tag{26}$$

where K_j^i and C_j^i are real constants such that $K_1^i > 0$ and $K_2^i < 0$ and $J_a(x)$ and $Y_a(x)$ are the Bessel functions of the first and second kinds respectively. The above equation of distribution is represented in Figure 5.

Figure 5. Representation of the population distribution of relation 26 where darker areas correspond to higher density. a) Depiction of the population density along the radius ρ for three instants of time $t_1 < t_2 < t_3$ (solid, dashed and dotted lines respectively) for $\sqrt{|K_2^i|} = 0$. In the remaining figures the

3D depiction is shown for different values of the constants. b: $\sqrt{|K_2^i|} = 0$, c: $\sqrt{|K_2^i|} = 1$, d: $\sqrt{|K_2^i|} = 2$, e: $\sqrt{|K_2^i|} = 3$.



Source: Author's representation

Both relations 21 and 26 share some important characteristics, namely the temporal behavior and the creation and regular arrangement of neighborhoods, although some difference between Figure 4 and 5 can be observed, for example, in Figure 5(d) and 5(e), the urban grid (the arrangement of city blocks) is no longer Hippodamian bun centralized or radial, which inherits the possibility of creating any number (even or odd) of main (equally important) local centers in a polycentric city. Indeed, in Figure 4(e) there can be exactly four main local centers, whereas by the application of relation 26, any number of main local centers can be created, as in Figure 5(d) (two centers) and in Figure 5(e) (three centers). It should be mentioned that the population distribution represented in all Figures 2, 3, 4 and 5, are chosen to be centrally symmetrical since, without any geographical influences, this is the best approximation of most common field cases but, by the appropriate choice of the constants C_m^i in relations 21 and 26, more irregular forms can be derived.

The heretofore presentation investigates only the building blocks (one dimensional inertial "system") P^i of the general system in relation 16, which, as mentioned in section 2, is derived by the transformation $Q^i = B_k^i P^k$, where B_k^i are real constants. This last equation is open to several interpretations depending on the observer's preferences, the first of which is a system of a single city inhabited by a group of N populations, each of which can represent a population class (economic, cultural etc.) or zone (habitation, commercial, industrial, common etc.). Hence, the equation of distribution of each population is: $Q^i(t,x,y) = B_k^i P^k(t,x,y)$ and the equation of distribution of total population of the city is given by the analytic expression:

$$Q(t,x,y) = \sum_{i=1}^{N} \left(Q^i(t,x,y) \right) = \sum_{k=1}^{N} \left(\sum_{i=1}^{N} \left(B_k^i \right) P^k(t,x,y) \right) \tag{27}$$

where the geographic coordinates (x,y) refer to the hypothetical city center $(0,0)$ and each $P^k(t,x,y)$ or $P^k(t,\rho,\theta\ )$ are special case of relations 21 or 26 respectively.

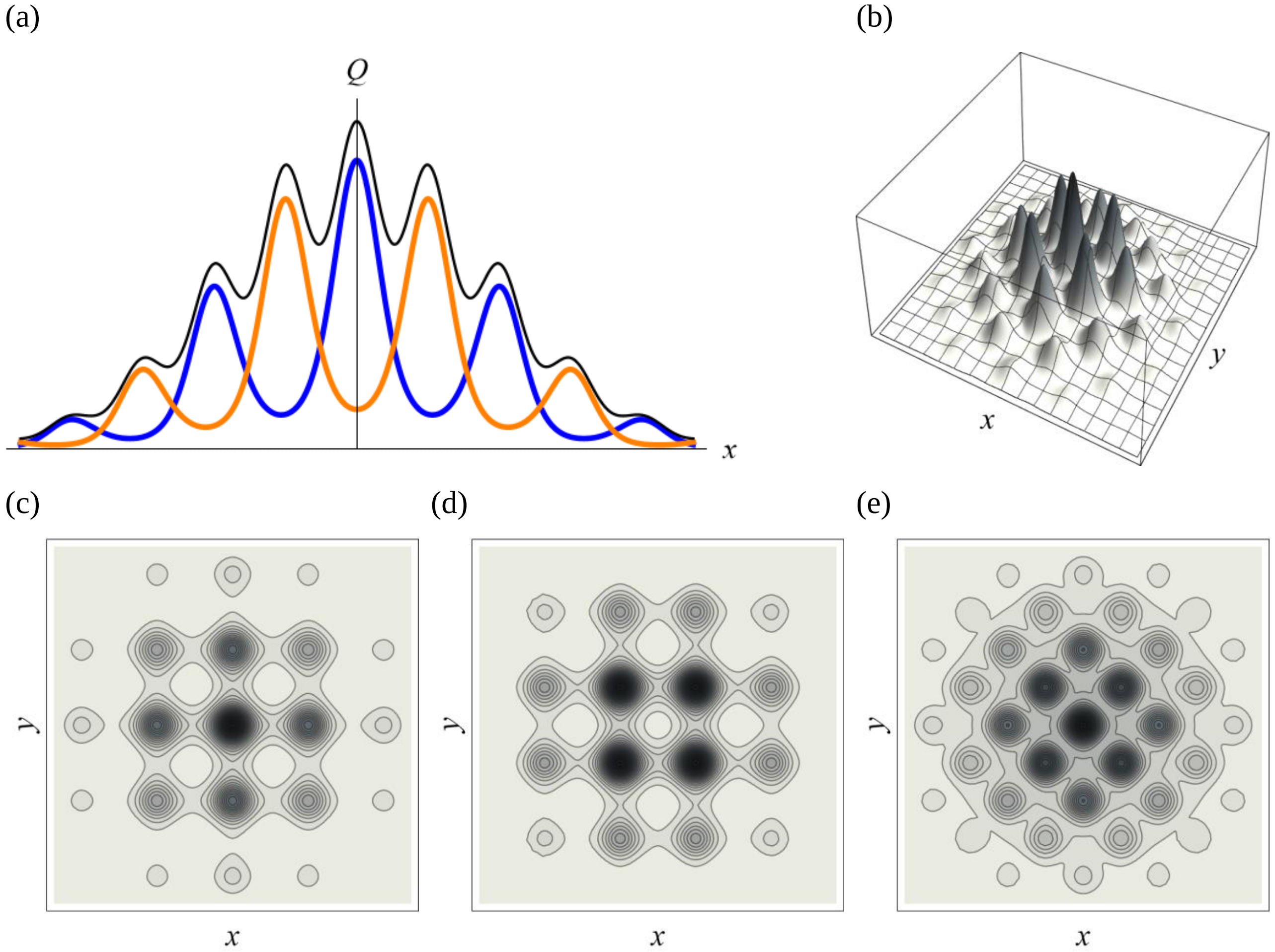
The linearity of relation 27 permits the deduction of the behavior of the city from that depicted in Figures 2, 3, 4 and 5, that is:

- On the early stages of the city, there is not distinguished geographical separation between the different population zones.
- As the city evolves, the geographical partition of the city becomes more and more prominent, forming interchangeable neighborhoods (areas) that are almost exclusively inhabited by a single population.
- The intensification of the partition leads to the formation of multiple administration (multiple municipalities) and functional (economic or habitation) centers, thus creating purely multicentric cities.

In Figure 6, the case of a single city, in advanced state of evolution, is depicted, consisting of two populations, namely Q^1 and Q^2, representing say two economic (income) classes of two zones (habitation and commercial), presented the partition of the geographic area. Figure 6 can be interpreted as describing one economic center (central pick of blue line, see Figure 6-d) and four municipalities (four picks of orange line, see Figure 6(e)). For reason of clarity, the distribution of each population is chosen to be as simple as possible and the geographical coordinates to be Cartesian, as in relation 21.

Figure 6. a: Representation of a single city consisting of two populations Q^1 (blue line) and Q^2 (orange line), so that the total population is $Q = Q^1 + Q^2$ (black line). b: 3D representation of the total population Q. c: Contour map of only the population Q^1. d: Contour map of only the population

Q^2. e: Contour map of the total population (union of the maps of Figures 6(c) and 6(d)). Darker areas correspond to higher density.



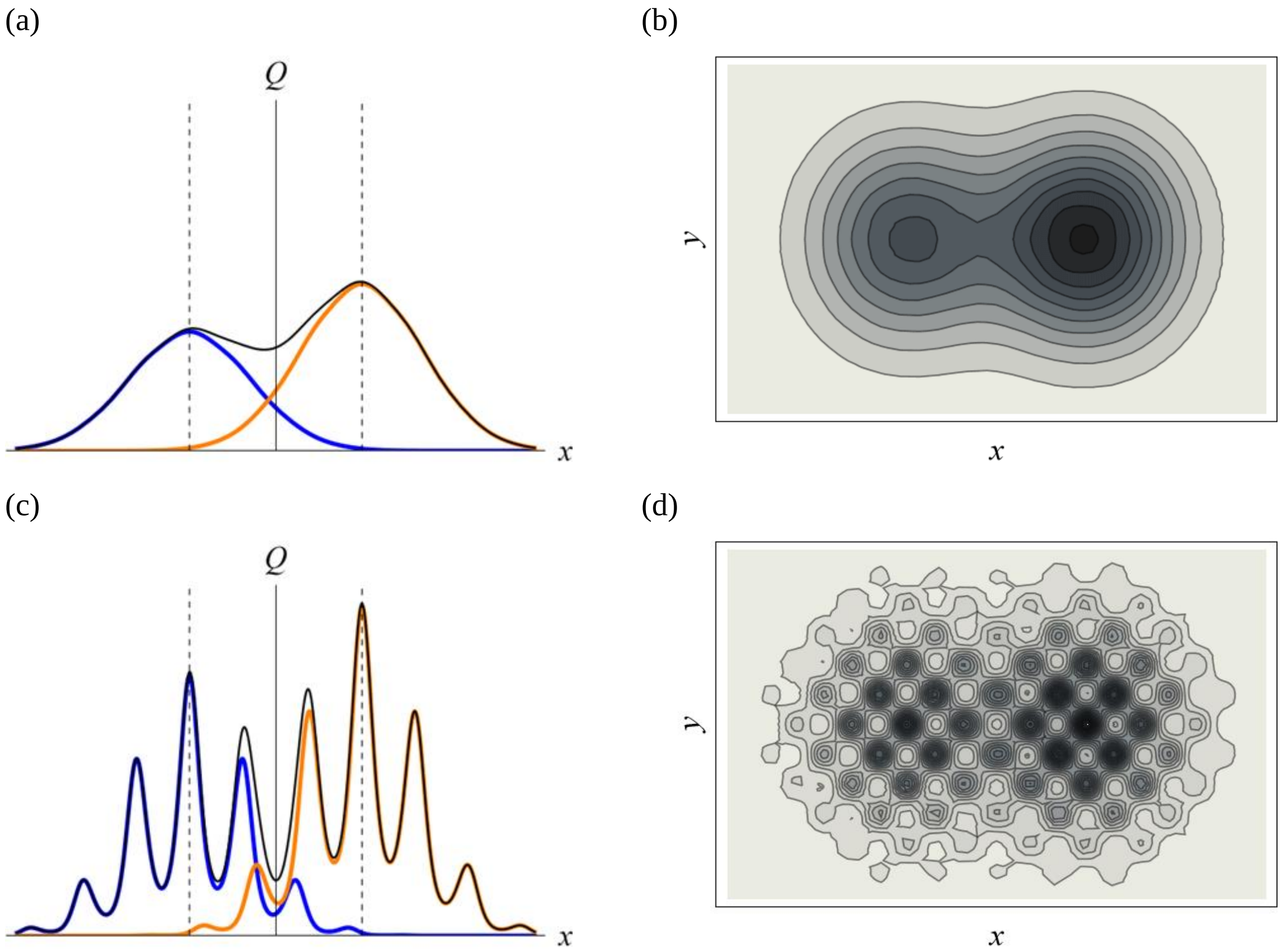
Source: Author's representation

Another interpretation of relation 16 is that of representing a system of multiple (N) cities where each Q^i stands for the total population of each city and the hypothetical city center of each one has geographical coordinates (X^i, Y^i). Then the equation of distribution of the system becomes:

$$Q^i(t, x - X^i, y - Y^i) = B^i_k P^k(t, x - X^i, y - Y^i) \quad (28)$$

where the coordinates (x, y) correspond to a randomly chosen "common center" of all cities and the functions $P^i(t, x, y)$ are given in relations 21 or 26. If the system is inertial or, at least its dynamic character is indetectable, its temporal behavior resembles that described in Figures 3, 4 or 5, namely, in the initial stages of the system the population density decreases monotonously as the distance from each city center increases and forms neighborhoods of high density in the advanced stage. In Figure 7, a system of two cities is depicted, such that both the city centers (dashed lines) are positioned along the x axis.

Figure 7. Representation of two comparable cities Q^1 (blue line) and Q^2 (orange line) and the total population of the system $Q = Q^1 + Q^2$ (black line). a: Early stage of the system, b: Contour map of Figure 7(a). c: Advanced stage of the system. d: Contour map of Figure 7(c). Darker areas correspond to higher density. The dashed vertical lines indicate the positions of the city centers. All figures have the same scale.

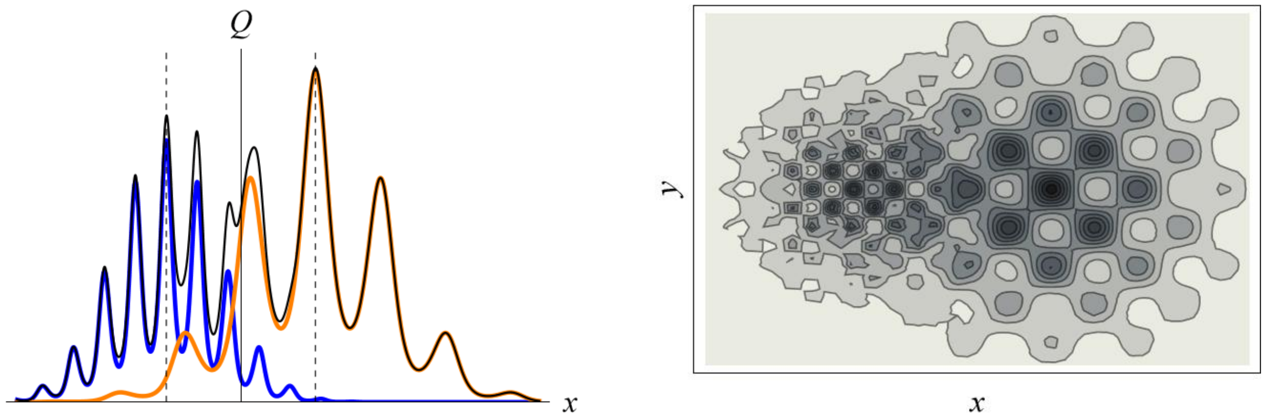


Source: Author's representation

It should be emphasized that Figures 7, especially Figure 7(d), representing relation 28, are simplified depictions of comparable cities, for reasons of clarity, so that the city blocks can be clearly distinguished and the infostructure (i.e. the transportation network) to be apparent, but in reality, as it can be shown in Figure 10, the neighborhoods of a system of multiple cities are not so regularly arranged. A system of disproportional cities, both quantitatively (size) and, more importantly qualitatively (wavelength and expansion) leads to irregularities along the border which is extended further into the area of the smaller city, as in Figure 8.

Figure 8. Representation of two disproportional cities Q^1 (blue line) and Q^2 (orange line) and the total population of the system $Q = Q^1 + Q^2$ (black line). a: Advanced stage of the system. b: Contour map of Figure 8(b), where darker areas correspond to higher density. The dashed vertical lines indicate the positions of the city centers.

(a) (b)



Source: Author's representation

Except for disproportionality, another reason for the irregularities at the inner arrangement of a city lies on the main hypothesis of the constitutional equations of the linear systems in relation 18 which is derived from relation 5. Indeed, it can be noticed that the condition for the population distribution to take the forms depicted in Figures 4, 5, 6 and 7, is that the geographical space, i.e. the habitat of the system to be flat, but, in reality, this is rarely the case. As can be seen in section 4, the shape of the city, that is the shape and the arrangement of the main city blocks, is uniquely determined by the topographical morphology of the habitat. A third reason is the burden of history and politics, which plays a profound role in shaping a city, since in the early stage of a city, the population distribution takes the form of Figures 3(c), 4(c), 5(b) or 6(e), that is, its behavior is mostly isotropic. The relatively low density, light traffic and the centralized socioeconomic state, favor the radial formation (see Figure 5), the Hippodamian formation of Classical Miletus being an exception rather than the rule, due to the destruction of the pre-Classical city and the special social structure of the Greek city-states. In a later period, the requirements originated from the high density and decentralization, lead to more orthonormal formations, as in Figures 3(e), 4(e), 6(e) or 7(d), hence the over time transition from polar to Cartesian formation, inevitably creates irregularities and disfunctions.

As mentioned in section 2, the dynamic character of a linear system represents the external action imposed on the system, the interaction between their components Q^i or both and is manifested by the existence of some complex eigenvalues λ^i (and their complex conjugates). If all the eigenvalues are complex (so that the dimension N of the system is an even number), then all the components P^i will take the form of relation 14 and they will behave according to Figure 1(a). Consequently, all the components Q^i will vanish periodically, as linear combinations of P^i, leading to the unviability of the system after some short period of time. To be precise, the system will not vanish, but it will collapse in its previous form. Indeed, at the moment that the first component, say Q^N, vanish, the dimension of the system will change to $N-1$ and the matrix of the system in relation 16 will be altered as a different $(N-1)\times(N-1)$ matrix from which new, odd in number, eigenvalues will derived, at least one of which will be real, thus transforming the resulting system to a viable one.

To demonstrate the behavior of a viable system, an example of a dynamic linear three-dimensional system (three cities) is presented, with the dynamic components being P^1 and P^2, and the cities being Q^i $(i=1,2,3)$. The equation of motion is derived by relations 12, 14 and 17 as:

$$Q^i(t) = B_1^i C^1 exp(\alpha^1 t)cos(\beta^1 t) + B_2^i C^2 exp(\alpha^1 t)cos(-\beta^1 t) + B_3^i C^3 exp(\lambda^3 t) \tag{29}$$

where $\lambda^1 = \alpha^1 + I\beta^1$, $\lambda^2 = \alpha^1 - I\beta^1$ and λ^3 are the eigenvalues of the system and B_k^i and C^i are real constants. The application of relation 13, 15 and 18 produces the equation of distribution:

$$Q^i(\boldsymbol{x}-\boldsymbol{x}^i) = B_1^i exp\left(U_{Re}^1(\boldsymbol{x}-\boldsymbol{X}^i)\right) cos\left(U_{Im}^1(\boldsymbol{x}-\boldsymbol{X}^i)\right) + B_2^i exp\left(U_{Re}^1(\boldsymbol{x}-\boldsymbol{X}^i)\right) cos\left(-U_{Im}^1(\boldsymbol{x}-\boldsymbol{X}^i)\right) + B_3^i exp\left(U^3(\boldsymbol{x}-\boldsymbol{X}^i)\right) \tag{30}$$

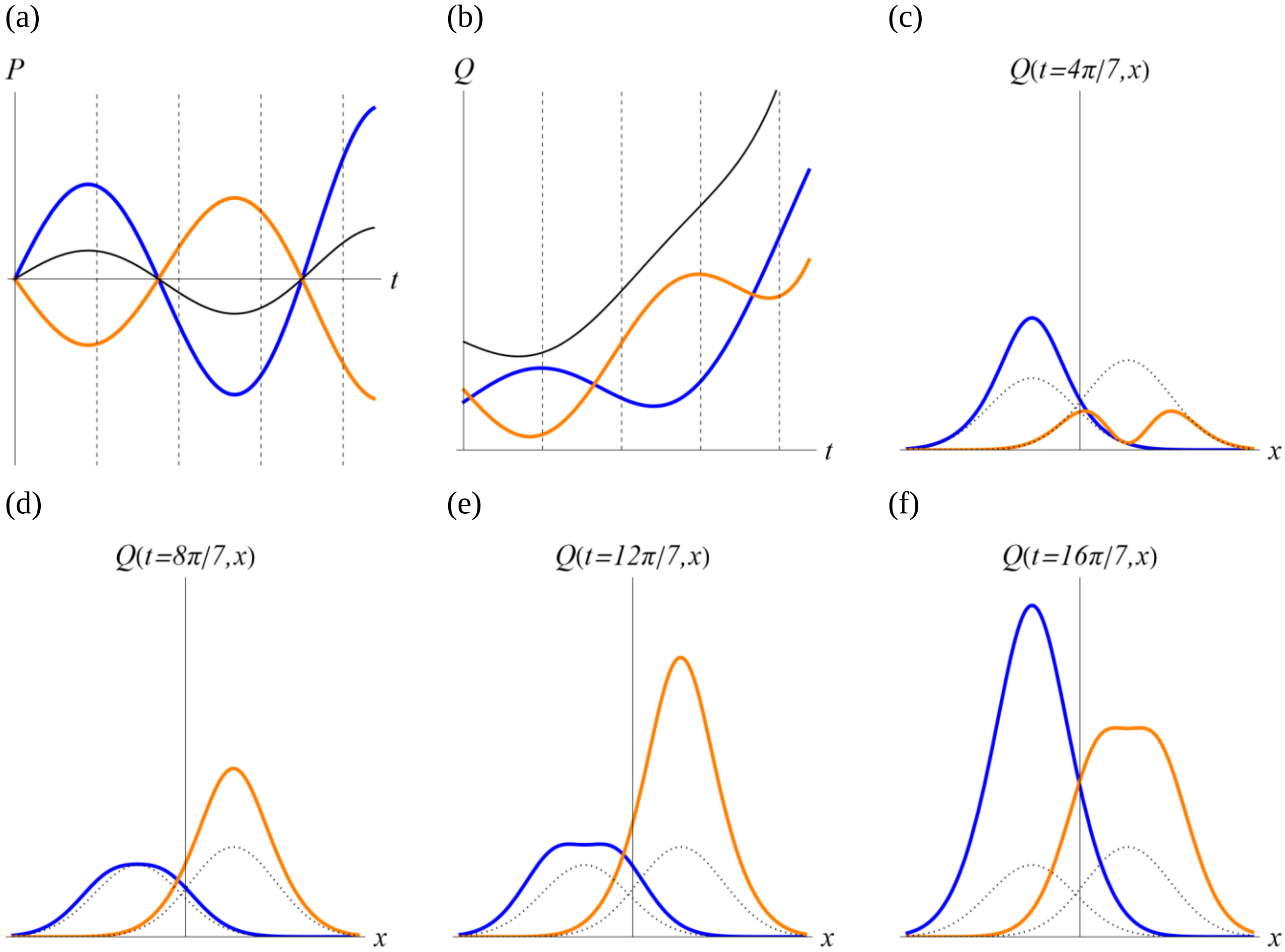
$$\textit{such that } \Delta U^i(\boldsymbol{x}) = 0$$

where $(\boldsymbol{x}-\boldsymbol{X}^i) \equiv (t, x-X^i, y-Y^i)$ are the geographical coordinates of the center of city i and the functions $U^i(\boldsymbol{x})$ are contained in relation 21 (or, for polar coordinates, in relation 26). In Figure 9 the behavior of relations 29 and 30 are depicted, with some simplifications of reasons of clarity:

- The two out of the three components are represented, Q^1 and Q^2, since the behavior of Q^3 is similar to the other two.
- Similarly, in Figure 9(a) only the dynamic components P^i are depicted, the third being an exponential function (see relation 30).
- In Figures 9(c), 9(d), 9(e) and 9(f), the distributions are simplified, using the depictions of Figure 7(a) rather than that of Figure 7(c), since the representation is focused on the initial stage of the cities.

Figure 9. Representation of relations 29 and 30, where P^1, Q^1 (blue line), P^2, Q^2 (orange line) and their sums (black) are depicted, at some geographical point. a: temporal behavior of the density of P^i. b: temporal behavior of density Q^i. Depiction of population distribution (along the axis x) of Q^i for four instants (marked by dashed lines in Figures 9(a) and 9(b)), all on the same scale. c: at $4\pi/7$.

d: at $8\pi/7$. e: at $12\pi/7$. f: at $16\pi/7$. The dotted lines of the diagrams c, d, e and f correspond to the time $t = 0$.



Source: Author's representation

The behavior or a dynamic system, described in Figure 9 can be described as follows:

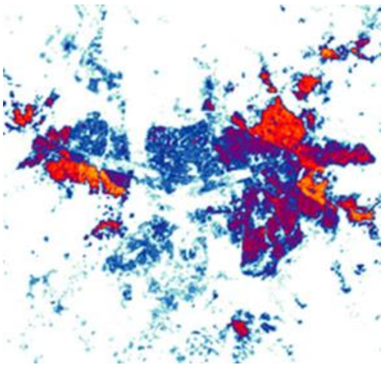
- Although the dynamic components P^1 and P^2 vanish periodically (Figure 9(a)), the system is sustainable, meaning that no component Q^i vanish, due to the existence of the inertial exponential component $P^i(t) = C^3 exp(\lambda^3 t)$ in relation 29 (not depicted in Figure 9).
- In the early stages of the development of the system, the trigonometric terms of relation 30 play a prominent role but as the system progresses, the importance of the periodic character diminishes due to the exponential function of P^3 as shown in Figure 9(b).
- Hence, the temporal behavior of a dynamic system at an advanced stage of development is difficult to distinguish from the exponential growth of an inertial system, at least within the margin of measurement error. That is why, as a rule, the regressions of statistical data (i.e. see Bergmann (2019)) and most population models (see the references in section 1) use some form

of monotonous exponential function to describe the population evolution, as in relation 25, although most cities constitute dynamic systems.

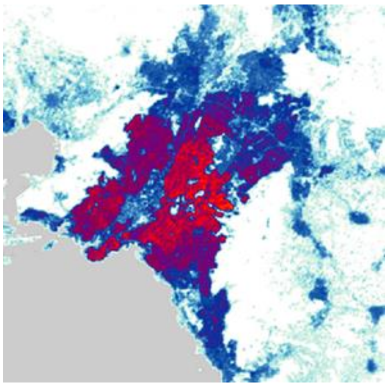
- A system of a single city with multiple populations (classes of zones) has an analogous temporal behavior like that of a system of multiple cities, as in Figure 9.

In Figure 10 the maps of some mainland cities are presented derived by Smith (2023), qualitatively comparable to some patterns examined in this section. Similar results, albeit not as extensive, can be found in Bergman (2019).

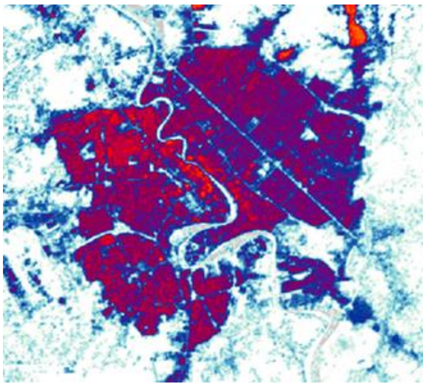
Figure 10. Maps of mostly inland cities around the World, qualitatively comparable to Figures 3, 4, 5, 6, 7 and 8, where sea (gray), land (white), lower density (blue), middle density (red) and high density (orange) are depicted.



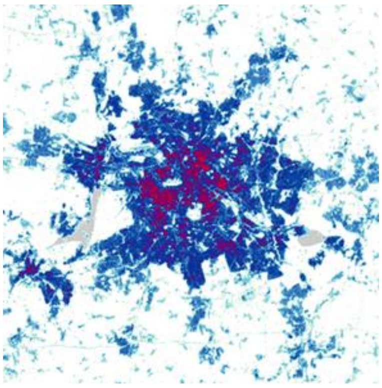
Ankara (Turkey)



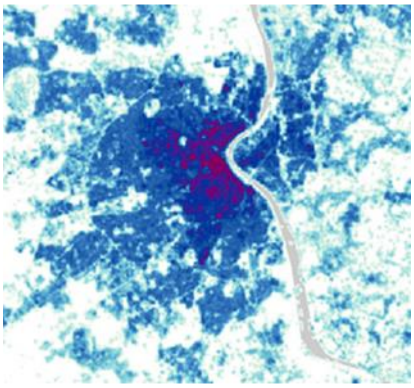
Athens (Greece)



Baghdad (Iraq)



Berlin (Germany)

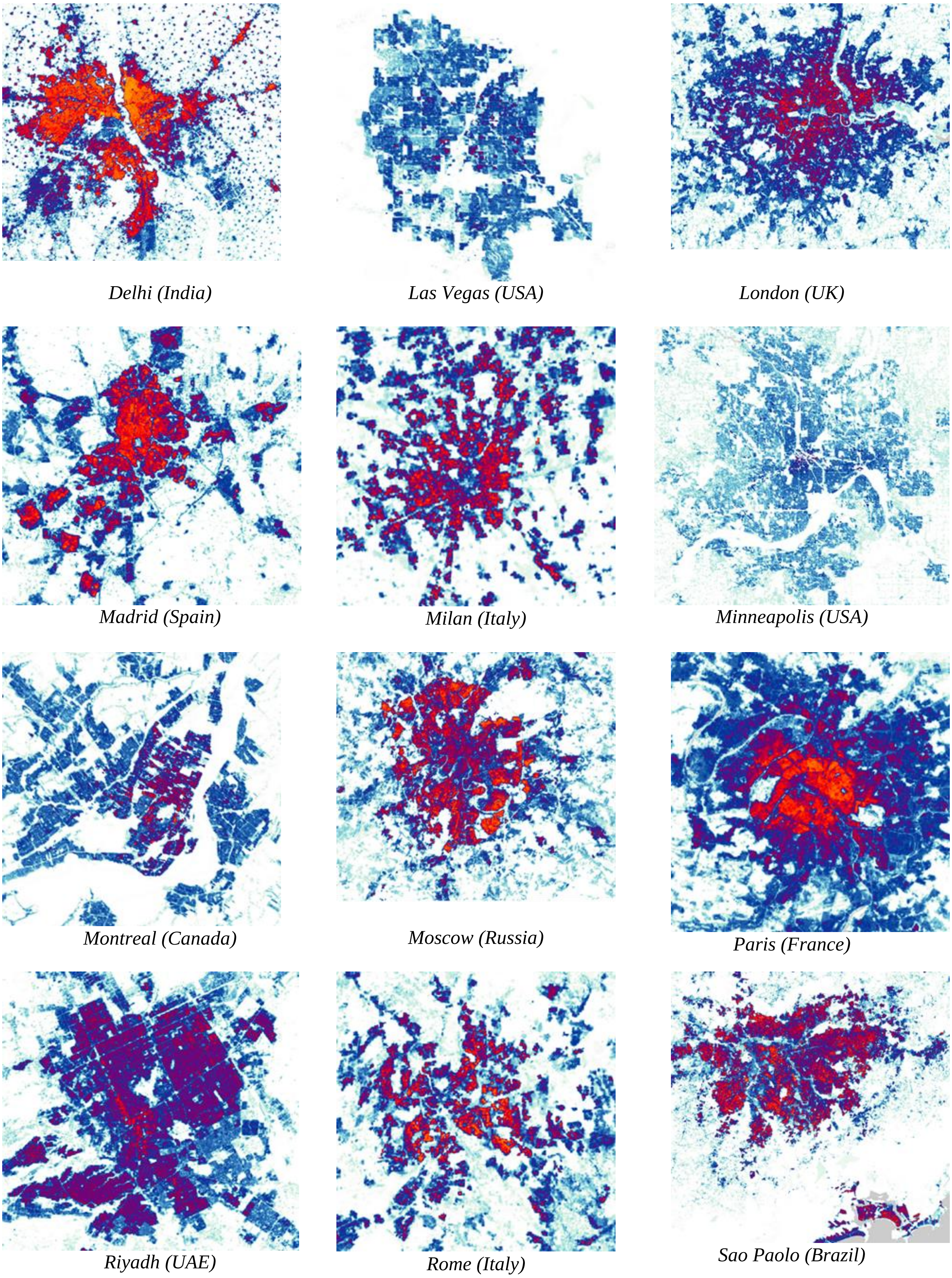


Bordeaux (France)

Colombus (USA)

Source: Smith D.A., CASA UCL (2023)

Figure 10. (continued)



Source: Smith D.A., CASA UCL (2023)

Some illustrative examples based on Figure 10 are the following:

- In relatively newly founded cities like Colombus, Las Vegas, Minneapolis, Montreal and Riyadh (where extensible reconstructions of the older town took place in 1950s) the application of the Hippodamian (orthonormal) grid can be observed as in Figures 3(b), 4(b) or 6(b). Also, in Las Vegas and Montreal polycentricity can be detected.
- In historical cities like Berlin, Bordeau and Paris, the almost monocentric radial grid appears, as described in Figure 5(b).
- Other historical cities like Delhi, London, Moscow and Rome present a polycentric radial grid as in Figures 5(d) and 5(e). The city of Madrid has also a polycentric radial grid but more similar to Figure 5(c), having a large center of gravity to the right and a number of disperse centers to the left of the picture.
- Systems of multiple historical cities like Ankara and Athens behave like Figure 7(d). The interesting cases of amalgamation of an old and new city, like Baghdad or Riyadh, can also be simulated as in Figure 7(d) with the distinction that in each case the old city follow the radial (left side of the picture in both cases) and the new city the Hippodamian (right side of the picture in both cases) grid.
- Megalopolises like Sao Paolo include great diversities within their population, incorporate many development plans for different areas of the city so that the best classification is that of a multiple city rather than that of a polycentric city.

A more detailed analysis and classification of a city requires the collection and processing of extensive statistical data about the density, not only of the total population but of their different components (classes or zones), for multiple points all over the city area and over a significant period.

4. Morphology of the habitat

Heretofore, the constitutional equation of distribution in relation 5 was constructed based in the assumption that the habitat of the system under examination does not affect the manner by which the population is distributed, so the spacetime of the habitat was considered as a Euclidean four-dimensional space and, moreover, the usual base of the Euclidian space $\boldsymbol{x} = \{x^0 \equiv t, x^1, x^2, x^3\}$, that is, time and the three Cartesian geographical coordinates, was sufficient for describing the behavior of the expansion of the system. This assumption facilitated the mathematical formulation of the model and, additionally, leads to a fairly adequate approximation of the behavior of many real-case population systems. The full generalization of the model for it to incorporate the effect of any topography of the habitat on the population system requires for the spacetime of the habitat to be not Euclidean but, in general, a Riemannian four-dimensional space having a usual base of the form: $\boldsymbol{w} =$

$\{w^0 \equiv t, w^1, w^2, w^3\}$ and a metric form, referred to the usual base: $(ds)^2 = \gamma_{\mu\nu} dw^\mu dw^\nu$, $\mu, \nu = 0,1,2,3$, where the components of the metric tensor are functions of the components of the base $\boldsymbol{w}$: $\gamma_{\mu\nu} = \gamma_{\mu\nu}(\boldsymbol{w})$. This formulation would lead to relation 5 of becoming overcomplicated, to the point that it would not be amenable to analytical solutions, even for the simplest cases of systems.

There are two tools by which the above situation can be manageable, the first of which is the simplification of the description of the habitat, by lessening the requirements for exact and/or extensive description of the topography. The spacetime of the habitat becomes a three-dimensional Euclidean space (time and two geographical coordinates) with the usual base $\boldsymbol{x} = \{x^0, x^1, x^2\}$, in which an admissible (continues and bijective) transformation to a curvilinear (still Euclidean) base $\boldsymbol{w} = \{w^0 \equiv t, w^1, w^2\}$ is applied, describing the variation of the topography. One further simplification is the consideration of this transformation to be orthogonal, so, finally, the simplified transformation takes the form of three bijective and continuous function of one variable: $t \equiv x^0 \equiv w^0$, $x^1 = x^1(w^1)$ and $x^2 = x^2(w^2)$. In this way, each geographical function can adequately describe the morphology of the habitat in a direction, along the longitudinal or the latitudinal line. One further calibration of the initial orientation of the base $\boldsymbol{x} = \{x^0, x^1, x^2\}$ can be made, for the longitudinal and latitudinal lines to include the major topographical features of the habitat.

The second simplification tool is the reduction of the constitutional equation of distribution in relation 5. The equation of distribution for every inertial system in general and for all linear systems, both inertial and dynamic, is given by relations 5 and 13 respectively, as functions of the usual base of the system: $Q^i(\boldsymbol{x}) = exp\left(U^i(\boldsymbol{x})\right)$, where the function $U^i = U^i(\boldsymbol{x})$ is a solution of the Laplace equation: $\Delta U^i(\boldsymbol{x}) = 0$, in the case of a flat habitat. Heretofore, for reason of brevity, both components Q^i or relation 10 and P^i of relation 13 will be symbolized as Q^i. Evidently, the information of the topographical variation (from the flat plane) of the habitat it should be inserted into the constitutional equations via the Laplace equation, by the transformation from the base $\boldsymbol{x}$ to the base $\boldsymbol{w}$. The result of the necessary calculations is the following differential equation:

$$\sum_{\mu=0}^{2}\left(\frac{\partial^2 U^i(\boldsymbol{x})}{\partial x^\mu \partial x^\mu}\right) = 0 \Rightarrow \frac{\partial^2 U^i(\boldsymbol{w})}{\partial w^0 \partial w^0} + \\ + \sum_{\mu=1}^{2}\left(\left(\frac{dx^\mu(w^\mu)}{dw^\mu}\right)^{-2}\frac{\partial^2 U^i(\boldsymbol{w})}{\partial w^\mu \partial w^\mu} - \left(\frac{dx^\mu(w^\mu)}{dw^\mu}\right)^{-3}\frac{d^2 x^\mu(w^\mu)}{(dw^\mu)^2}\frac{\partial U^i(\boldsymbol{w})}{\partial w^\mu}\right) = 0 \tag{31}$$

which is a reductive form of the Laplace – Beltrami equation (see for example Besse (1987) or Taylor (2011)). In this section the above relation is applied to two general examples, namely for cities built on uniformly inclined terrain and for cities constructed near the coastline (or any impenetrable barrier).

The uniform inclination of the terrain can be simulated by the transformation:

$$x^0 = w^0 \equiv t, x^1 = (A+1)w^1, x^2 = w^2 \tag{32}$$

where $A \geq 0$ is real constant corresponding to the slope of the terrain. Hence, relation 25 becomes:

$$\frac{\partial^2 U(\boldsymbol{w})}{\partial t \partial t} + \frac{1}{(A+1)^2}\frac{\partial^2 U(\boldsymbol{w})}{\partial w^1 \partial w^1} + \frac{\partial^2 U(\boldsymbol{w})}{\partial w^2 \partial w^2} = 0 \tag{33}$$

a general solution of which is:

$$\begin{aligned} U(\boldsymbol{w}) = &\left(C_1 exp\left(t\sqrt{K_1}\right) + C_2 exp\left(-t\sqrt{K_1}\right)\right)\left(C_3 exp\left((A+1)w^1\sqrt{K_2}\right)\right. \\ &\left. + C_4 exp\left(-(A+1)w^1\sqrt{K_2}\right)\right)\left(C_5 exp\left(w^2\sqrt{K_3}\right) + C_6 exp\left(-w^2\sqrt{K_3}\right)\right) \\ &+ \frac{K_4}{2}(t)^2 + \frac{(A+1)^2 K_5}{2}(w^1)^2 + \frac{K_6}{2}(w^2)^2 + C_7 t + C_8 w^1 + C_9 w^2 + C_{10} \\ &K_1 + K_2 + K_3 = 0\,, K_4 + K_5 + K_6 = 0 \end{aligned} \tag{34}$$

where K_i and C_i are real constants (different for each relation).

To specify the population distribution, some conditions concerning relation 34 could be considered:

- If $U(\boldsymbol{w})$ is a solution of relation 34, the equation of distribution referred to the natural (population) base is given by the transformation: $Q(\boldsymbol{w}) = exp\left(U(\boldsymbol{w})\right)$ of relation 10.
- In the cased of $A = 0$ in relations 32, 33 and 34 (no inclination of the terrain), the distribution of the population should behave in the same way to both directions w^1 and w^2. Therefore, in relation 34: $K_2 = K_3 = -K_1/2$ and $K_5 = K_6 = -K_4/2$
- Any function of time t must be uniformly monotonous, hence: $K_1 > 0$. Moreover, the population density cannot increase indefinitely from the center outward, hence $K_4 > 0$.

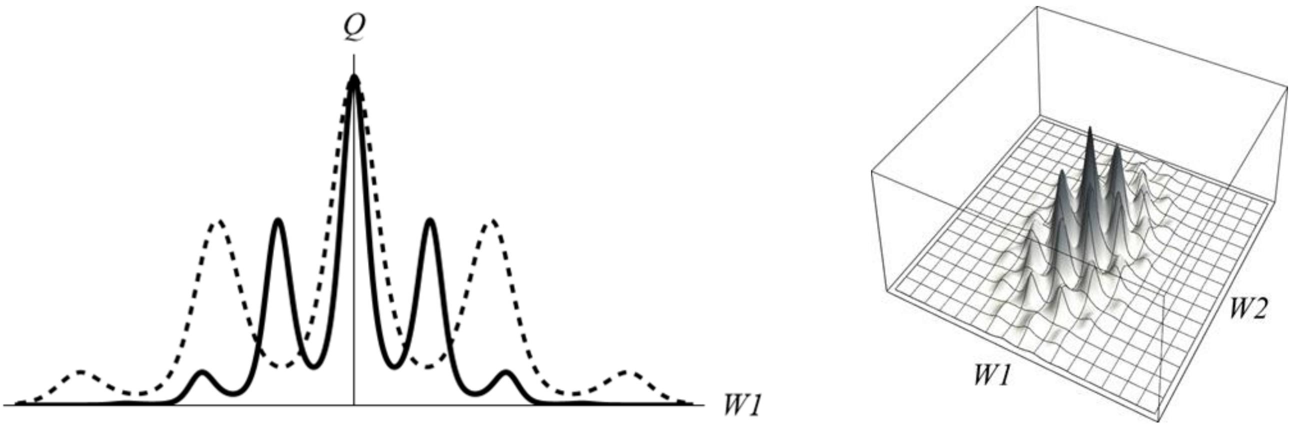
The equation of distribution referred to the natural base becomes, after some rearrangement:

$$\begin{aligned} Q(\boldsymbol{w}) = exp&\left(C_1 exp\left(t\sqrt{K_1}\right) cos\left((A+1)w^1\sqrt{|K_1|/2} + C_2\right) cos\left(w^2\sqrt{|K_1|/2} + C_3\right)\right. \\ &\left. + \frac{K_4}{2}(t)^2 - \frac{(A+1)^2 K_4}{4}(w^1)^2 - \frac{K_4}{4}(w^2)^2 + C_4 t + C_5 w^1 + C_6 w^2 + C_7\right) \end{aligned} \tag{35}$$

where C_i and $K_i > 0$ are real constants. It can be mentioned that, in case of $A = 0$, relation 35 describes a bidirectional horizontal terrain. Relation 35 is depicted in Figures 11 and 12.

Figure 11. Representation of relation 35 for a constat value of time. a: Population distribution along the axis w^1 for $A > 0$ (solid line) and $A = 0$ (dashed line). b: Three-dimensions graph of Figure 11(a).

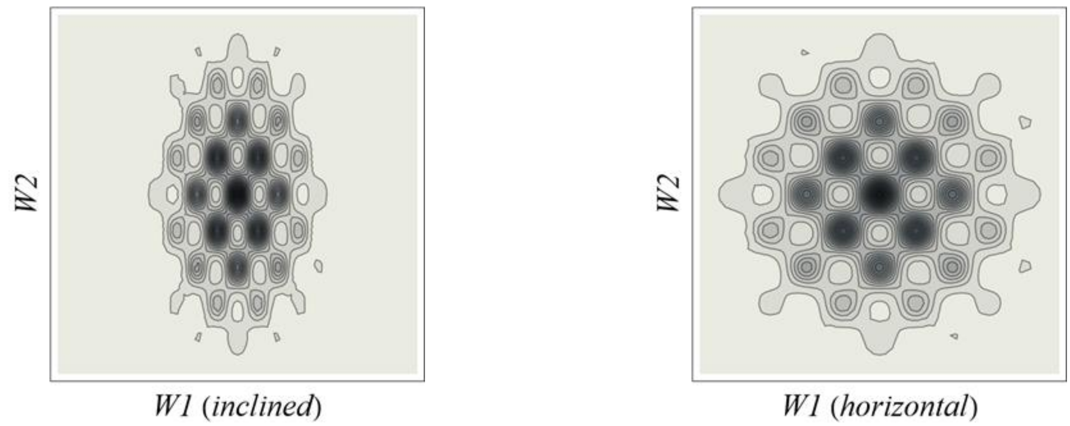
(a) (b)



Source: Author's representation

Figure 12. Contour plot for the population distribution of Figure 11(b), where darker areas represent higher density. a: $A > 0$. b: $A = 0$.

(a) (b)



Source: Author's representation

The next example deals with the existence of an impenetrable barrier (i.e. shoreline) beyond which no population can expand. An appropriate transformation should have the following characteristics:

- It can be assumed, without a loss of generality, that this barrier occurs at the directions of the axis x^1, hence the transformation can be given by an acceptable (reversible) function $x^1 = x^1(w^1)$.
- The domain of this function should be: $w^1 \in [-L, +\infty[$, where L is the (positive) distance of the barrier from the hypothetical center of the city and its range: $x^1 \in]-\infty, +\infty[$ (see Figure 14-left).

- The function should be simple enough for relation 31 to produce an analytical solution.

One of the many possible transformations satisfying the above conditions is the following:

$$x^0 = w^0 \equiv t, x^1 = A ln(w^1 + L), x^2 = w^2 \tag{36}$$

which transforms relation 31 to:

$$\frac{\partial^2 U(\boldsymbol{w})}{\partial t \partial t} + \frac{(w^1 + L)^2}{A^2}\frac{\partial^2 U(\boldsymbol{w})}{\partial w^1 \partial w^1} + \frac{w^1 + L}{A^2}\frac{\partial U(\boldsymbol{w})}{\partial w^1} + \frac{\partial^2 U(\boldsymbol{w})}{\partial w^2 \partial w^2} = 0 \tag{37}$$

The general solution of relation 37 is given by:

$$\begin{gathered} U(\boldsymbol{w}) = \left(C_1 exp\left(t\sqrt{K_1}\right) + C_2 exp\left(-t\sqrt{K_1}\right)\right)\left(C_3 cosh\left(A\sqrt{K_2} ln(w^1 + L)\right)\right. \\ \left. + iC_3 sinh\left(A\sqrt{K_2} ln(w^1 + L)\right)\right)\left(C_5 exp\left(w^2\sqrt{K_3}\right) + C_6 exp\left(-w^2\sqrt{K_3}\right)\right) \\ + \frac{K_4}{2}(t)^2 + \frac{\left(C_7 + (A)^2 K_5 ln(w^1 + L)\right)^2}{2(A)^2 K_5} + \frac{K_6}{2}(w^2)^2 + C_8 t + C_9 w^2 + C_{10} \\ K_1 + K_2 + K_3 = 0\,, K_4 + K_5 + K_6 = 0 \end{gathered} \tag{38}$$

where K_i and C_i are real constants. As in the previous example, some constraints should be imposed to the final form of the equation of distribution:

- For an inertial system, for every point of the habitat the equation of distribution should be a uniformly monotonous function of time, hence: $K_1, K_4 > 0$.
- The population density should not continuously increase outward, hence: $K_5, K_6 < 0$.

The equation of distribution is derived from 38 as:

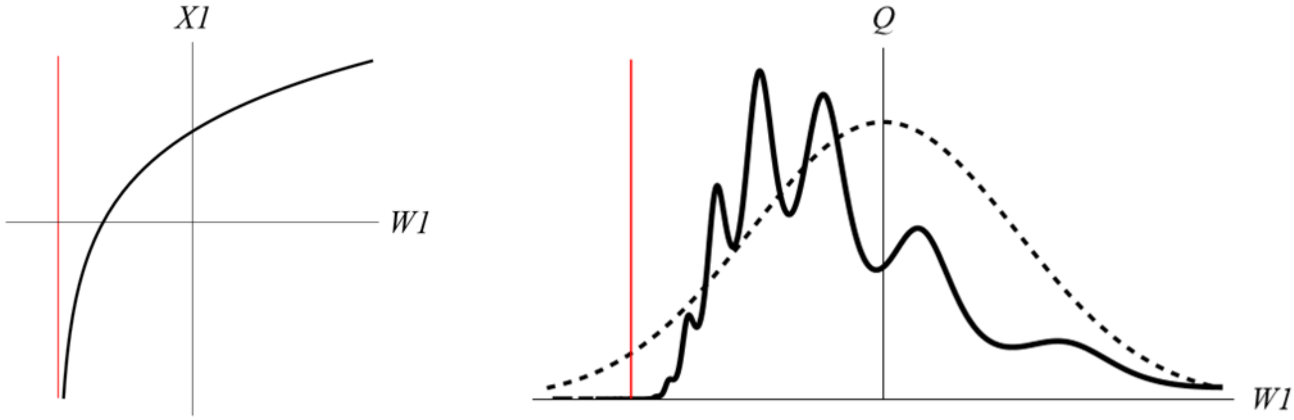
$$\begin{gathered} Q(\boldsymbol{w}) = exp\left(\left(C_1 exp\left(t\sqrt{K_1}\right)\right) cos\left(A ln(w^1 + L)\sqrt{|K_2|} + C_2\right) cos\left(w^2\sqrt{|K_3|} + C_3\right)\right. \\ \left. + \frac{K_4}{2}(t)^2 + \frac{\left(C_4 + (A)^2 K_5 ln(w^1 + L)\right)^2}{2(A)^2 K_5} + \frac{K_6}{2}(w^2)^2 + C_5 t + C_6 w^2 + C_7\right) \end{gathered} \tag{39}$$

In Figures 13 and 14, relation 39 is represented. In Figure 15 some large coastal cities are sampled, the behavior of which, pending terrain irregularities, resembles that of Figure 14, that is, high population density exists near the shoreline, the population density decreases towards the mainland and sporadic areas of higher density around the main concentration can be noticed. Also, as in the depictions of section 3, the population is regularly arranged in alternative areas of high and low density (neighborhoods) through the habitat. According to the boundary conditions of the simulation, smooth shoreline and terrain were chosen.

The mathematical simulation of the topography of the habitat (as a whole or in parts), as in relation 32 or 36 and, consequently, the derivations of the elementary population distributions, as in relations 35 or 39 respectively, can immediately determine the behavior of any linear system expanded on an anomalous territory, by corresponding relations 35 or 39 to the building blocks $P^i(\boldsymbol{x})$ in relation 18. It should emphasize the limitations of the mathematical model presented in this section, leading to relation 31, that is the condition that it can be applied to general inertial systems and to linear systems both inertial and dynamic. Furthermore, the functions by which the topography of the habitat is simulated should be continuous and bijective and, also to be as simple as possible, for the differential equation of relation 31 to be amenable to analytical solution.

Figure 13. a: Representation of the transformation of relation 36 (Black line) and the barrier at $w^1 = -L$ (red line). b: Representation of the population distribution of relation 39 (black solid line), the barrier (red line) and the enveloping curve of the population distribution in the case of no barrier (black dashed line).

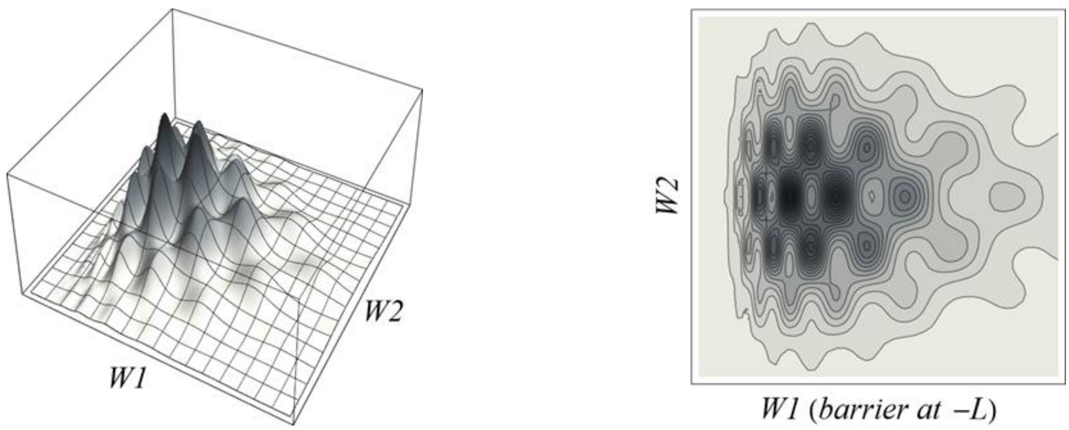
(a) (b)



Source: Author's representation

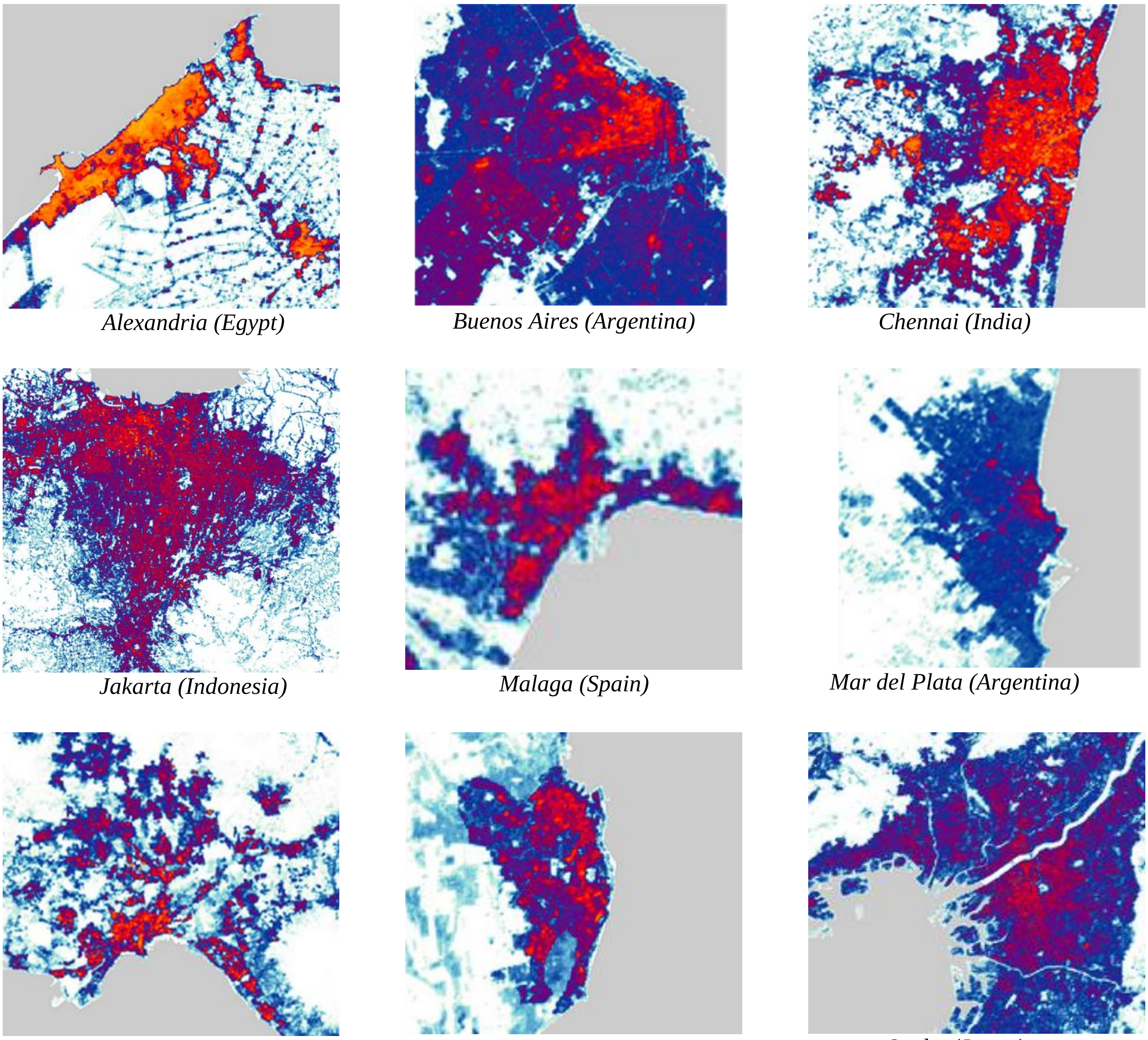
Figure 14. Representation of Figure 13(b), where darker areas represent higher density. a: Three-dimensional plot. b: Contour map.

(a) (b)



Source: Author's representation

Figure 15. Maps of coastal cities around the World, qualitatively comparable to Figure 14, where sea (gray), land (white), lower density (blue), middle density (red) and high density (orange) can be observed.



Alexandria (Egypt) — *Buenos Aires (Argentina)* — *Chennai (India)*

Jakarta (Indonesia) — *Malaga (Spain)* — *Mar del Plata (Argentina)*

Naples (Italy) — *Odesa (Ukraine)* — *Osaka (Japan)*

Source: Smith D.A., CASA UCL (2023)

5. Concluding remarks

The fundamental element for the derivation of the equation of distribution of an inertial system or, as in the case the presents paper, of a linear system, both inertial and dynamic, is Laplace equation, applied to the usual base of the system, as indicated in relations 6 and 13. Since Laplace equation is a linear, elliptic, differential equation, for its solutions the principle of superposition is valid:

$$\Delta U_a(\boldsymbol{x}) = 0 \Rightarrow \Delta\left(\sum_{a=1}^{A} C^a U_a(\boldsymbol{x})\right) = 0 \tag{40}$$

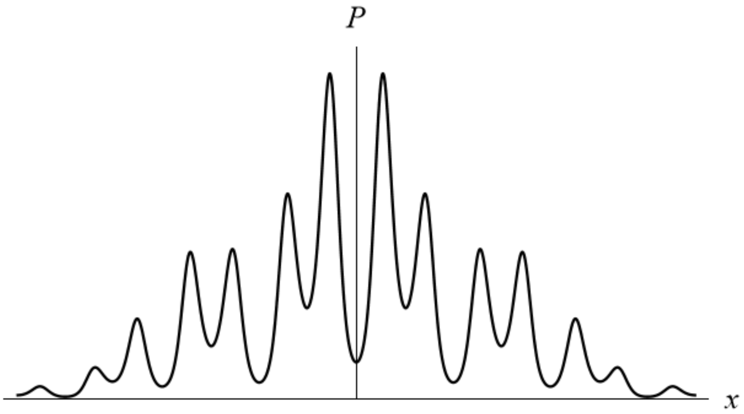
where C^a are real constants and A is positive integer. Relation 40 permits the generalization of the elementary solutions of relations 21 and 26 as:

$$P^i(\boldsymbol{x}) = exp\left(\sum_{a=1}^{A} C^a U_a(\boldsymbol{x})\right) : \ \Delta U_a(\boldsymbol{x}) = 0 \qquad (41)$$

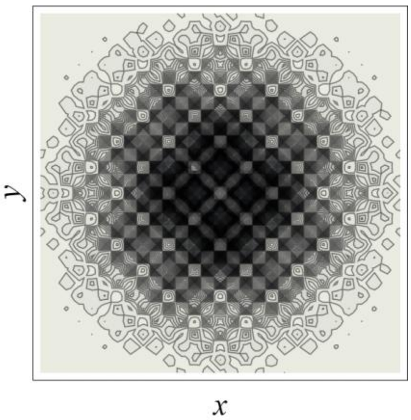
The result of this generalization leads the population distribution to a far more complex and interesting behavior. For example, in Figure 16, the generalization of relation 21 is depicted for $A = 3$, where Figure 16(b) provides more details and insight about the character and the structure of the city, than the corresponding Figure 3(b).

Figure 16. Representation of the application of superposition to relation 21 for a constat value of time. a: Population distribution along the axis x. b: Contour map of Figure 16(a).

(a)



(b)



Source: Author's representation

Since the analytical model presented in Elias (2023) and summarized in section 1 was built by using only three fundamental axioms, common for Physical systems, it can be assumed, at least in theory, that the deduced equations of distributions provide the "natural' way by which a population settles and expands, that is "from the bottom up". An attempt to circumvent this natural distribution by some "from the top down" implementation can easily lead to discomfort, disfunctions or, even to failure, hence any model that provides such natural distribution can operate as an optimal control modulus for any administration planning, from the optimization of transportation network and zoning to the prediction of the behavior of inner cities. The existence of analytical equations of distributions can facilitate any optimization attempts by directly applying well-known methods of calculus of variations to the existing equations.

The linear population systems with constant coefficients provide the perfect balance between approximation of the real phenomenon and the volume of computations required for any mathematical model to work. One significant benefit of linearity is that it facilitates the expansion of the model to include more cities of classes of populations, by the implementation of the simple addition operation. In the author's opinion, the main disadvantage of linear systems is their inner difficulty in properly expressing the dynamic state. As presented in section 2, by the construction of linear systems, the dynamic influence alters the space of reference from Euclidean (inertial state) to Pseudo-Euclidean (dynamic state), instead of Riemannian, as in more general systems (see section 1). Consequently, the constitutional equations include only trigonometric or products of trigonometric and exponential dynamic terms, leading the dynamic system to a binary behavior, either to vanish in a short period or, after some time, to grow exponentially (as if it was an inertial system), as described in relation 30 and Figure 9.

A more comprehensive perception of the dynamic behavior of the population can be achieved by the study of more general, non-linear systems. A class of such systems that presents analytical interest and practical importance is that of systems whose dynamic influence leads to sustainable behavior, that is of systems being in equilibrium with their environment.

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