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A MATHEMATICAL MODEL FOR POPULATION DISTRIBUTION

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Biographical Note

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Abstract

In the present paper, an attempt is made to construct a deterministic mathematical simulation for population systems, by which their temporal (equation of motion) and spatiotemporal (equation of distribution) behaviour can be deduced, as solutions of the constitutional differential equations of the system. The generic formulation of the constitutional equations gives the simulation the possibility to expand to several populations, but also to parameters of different nature (say economic), by applying proper transformations according to the inner properties of each parameter. The introduction of the topographical features of such a system can be reduced to a boundary conditions problem, applied to the constitutional differential equations. Two initial applications are analyzed herein, namely a one-dimensional inertial population system, and a one-dimensional dynamic population system, where the external force corresponds to a space of constant curvature. The theoretically predicted behaviors of the population distribution of these systems are compared qualitatively to actual field data, collected from cities around the World.

Keywords: mathematical simulation, constitutional equations, population distribution.
JEL Classification: Y80

1. Introduction

There are many equations describing the population distribution of towns, cities or metropolitan areas, which relate the population density to the polar radius from the city center, i.e., the hypothetical or actual birth point of the city. One of the most recognized of these, is the equation suggested by Clark (1951), which provides both directness and experimental (statistical) validation:

$$P(\rho) = P(0)exp(A\rho) \tag{1}$$

where P is the population density at a given radius $\rho \geq 0$ from the center of the city, the quantity $P(0) \equiv P(\rho = 0)$ in relations (1) (2) and (3) is the value of the population density of the city at the city center and $A < 0$ is a real constant estimated empirically. This equation provides both simplicity of application (only two constants must be estimated from the statistical data) and experimental validation. Since then, the accumulation of statistical data has led to variations of relation (1), such as the modified Zielinski – Frankena equation in Martori and Surinach (2001)):

$$P(\rho) = P(0)exp\big((A\rho + B\rho^2 + C\rho^3)\rho^D\big) \tag{2}$$

where A, B, C and D are real constants estimated empirically. Most of these equations have the following general form:

$$P(\rho) = P(0)exp\left(\sum_{m=0}^{M}(C_m\rho^m)\right) \tag{3}$$

where m and M are integers and C_m are real constants estimated empirically, as presented in Zielinski (1980), Anselyn and Can (1986), Martori and Surinach (2001) or Griffith and Wang (2007). An extensive work by Bertaud and Malpezzi (2003) with the cooperation of the World Bank, includes the field data from forty-eight major cities around the World in the form of graphs (density versus radius) and approximates the data with an exponential function as in relation (1). Some of these graphs are included herein, as a qualitative correlation between the field data and the theoretically derived equations.

Recent models proposed various equations for population density as a function of the polar radius: $P = P(\rho)$, some of which are special cases of relation (3). An extensive survey with field data collected from the fifty largest cities of the world (comparable to the work of Bertaud and Malpezzi) made by Subasinghe, Wang and Murayama (2022), proposed the categorization of the above cities into five distinct categories with respect to the following equations:

$$\begin{gathered} P(\rho) = Aexp(B\rho) \\ P(\rho) = A(\rho)^B \\ P(\rho) = Aexp\left(-\left(\frac{\rho - B}{C}\right)^2\right) \end{gathered} \tag{4}$$

where A, B and C are real constants estimated empirically (the first equation includes three categories with different values of the real constants).

Other works, instead of performing a global survey, focus on a specific area, including one of several nearby cities, but collect the data of the same research subject at different instants of time, although the approximating equations do not include the variable of time, but vary the constants for each instant. The work by Khatun, Falgunee and Kutub (2015), study the Dhaka Metropolitan Area in 2001 and 2011 by proposing the following equations:

$$P(\rho) = Aexp(-B\rho)$$
$$P(\rho) = -Aln(\rho) + B \qquad (5)$$
$$P(\rho) = -A(\rho)^2 + B\rho + C$$

where A, B and C are real constants estimated empirically. Also, Feng and Chen (2021) study the city of Hangzhou, China in the years 1964, 1982, 1990, 2000 and 2010 by using three of the equations proposed by Khatun, Falgunee and Kutub (2015) and adding three new:

$$P(\rho) = -A\rho + B$$
$$P(\rho) = Aexp(-B\rho)$$
$$P(\rho) = -Aln(\rho) + B$$
$$P(\rho) = -A(\rho)^{-B} \qquad (6)$$
$$P(\rho) = Aexp\left(-B\big(ln(\rho)\big)^2\right)$$
$$P(\rho) = Aexp(-B(\rho)^C)$$

where A, B and C are real constants estimated empirically. For the study of ten cities of Liaoning Province, China, Guo and Wang (2022), introduce the inverse sigmoid equations:

$$P(\rho) = \frac{1 - A}{1 + exp\left(B\left(\frac{2\rho}{C} - 1\right)\right)} + A \qquad (7)$$

where A, B and C are real constants estimated empirically. A more holistic survey is given by Bergmann (2019), in the sense that it contains thirteen large cities around the World (Europe, America, Asia) and for each, the field data are given (density versus polar radius) for four instants, in the years 1975, 1990, 2000 and 2015. The data are approximated by the Clark equation (1). In all the above cases, one can observe that for the same research subject (city), the approximating equation differs for every instant of time, quantitatively (same equation with different constants) or even qualitatively (altogether different equation) to counterbalance the absence of time as an independent variable.

An equation producing the population density as a function of the polar radius, as in relation (2) and all the others mentioned above, incorporates some presumptions concerning the phenomenon of population distribution, that may facilitate the construction of a functional model, but can, occasionally, present some limitations for understanding its behavior:

- The independence of the population density equation from time, limits the possibility of evolutionary study of the phenomenon, since the homogeneous increase of the population density at every point of a city, following a past pattern, cannot be always the case.

- By the dependence of the equation on the polar radius only, an assumption has been made of an isotropic propagation of the phenomenon, which is only valid in the case of the absence of directed significant external influences, that is, geographic variations, resources, other cities, etc. Even in the case of the absence of external influences, the behavior of a homogeneous distribution should be examined without the presumption of isotropy.
- A statistically derived equation cannot be easily amenable to qualitative analysis, nor can be expanded to include external influences, multiple populations, or variables of different nature, say various demographic of economic parameters.

The above concerns lead to the pursuit of some function of the general form:

$$P = P(t, x, y, z) \rightarrow P(t, x, y) \rightarrow P(t, \rho) \tag{8}$$

which can be reduced to a function $P = P(\rho)$, only as a special case and for a specific instant in time. Furthermore, an attempt has been made to construct a mathematical model which could analytically deduce such equations by using the minimum number of axiomatic principles. This paper does not present such a complete model, only some essential parts which can lead to generalization, that is the derivation of the constitutional equations, its solutions in the case of one-dimensional inertial population system, a notion of interacting population systems and a special case of a dynamic system, namely one influenced by a force of constant curvature. Apart from papers relevant to Regional Science (some of which are mentioned above), principles of Classical and Relativistic Mechanics were used, albeit applied to a different space of reference. Indeed, all the important objects needed for the creation of a population model were treated here: the events, the path, the state and the action. Some of the textbooks used from the latter category include Gelfand and Fomin (1963), Smirnov (1964), Landau and Lifshitz (1980), Dodson and Poston (1997), Francoise, Nabel and Tsun (2006), Itskov (2007), Talman (2007) and Bourles (2019).

2. Description of the general model

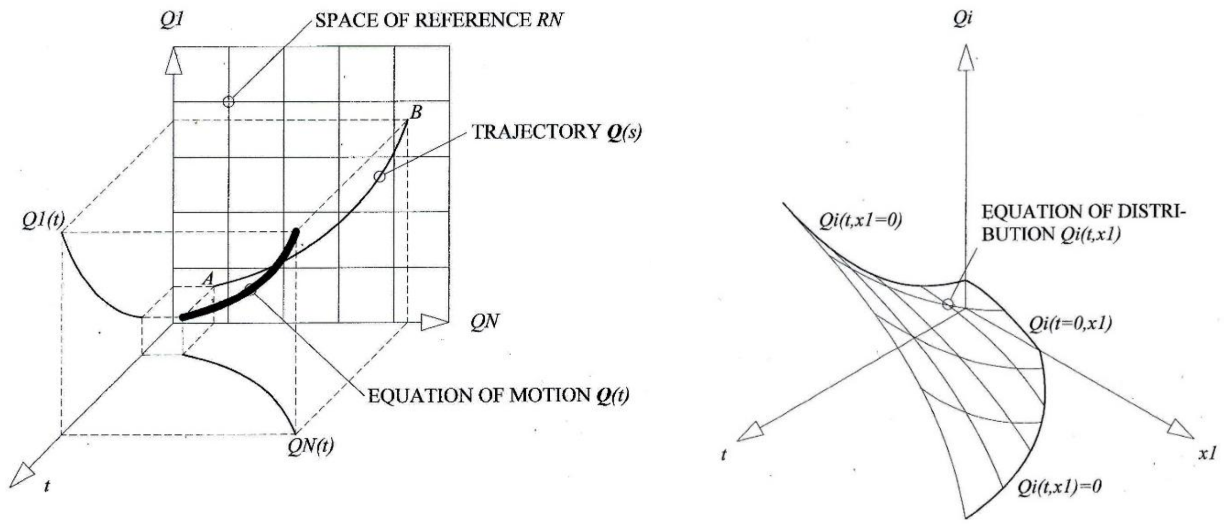
In creating the space of reference of the model, one can primarily observe that the geographic space and time of Classical of Relativistic Mechanics is not suitable for this purpose, since some variation of the spatial or temporal coordinates cannot, by themselves, cause any variation of the system. Indeed, in the case of a truly isolated city, its location and foundation time do not affect the character of its evolution and expansion, as long the morphological and climatological conditions remain the same. As for these conditions, they are independent of the geographical coordinates and can be introduced to the constitutional equations as a boundary value problem. On the contrary, the parameters mentioned above, that is populations, demographics, economy, etc., do affect the system, hence, some of them could constitute the space of reference of the model.

Although the number and the nature of these parameters are not yet determined, one can assume that there exists a set of N such parameters that are linearly independent to each other and, at the same time, they can adequately describe the behavior of the system, say: $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$. Such a set can be one of the bases of an N-dimensional Riemannian (in general) space $\mathcal{R}_N$ which has a metric form:

$$(ds)^2 = g_{ij}(Q^1, \dots, Q^N) dQ^i dQ^j \ , i, j = 1, \dots, N \tag{9}$$

where ds is the elementary arc length of any line imbedded into $\mathcal{R}_N$ and $g_{ij} = g_{ij}(Q^1, \dots, Q^N)$ are the $N \times N$ components of the metric tensor for the base $\boldsymbol{Q}$.

Figure 1. Representation of the essential objects of the model.



Source: Author's representation

By adopting $\mathcal{R}_N$ as the space of reference of the system, every event $\boldsymbol{Q} = (Q^1, \dots, Q^N)$ of the system corresponds to a point of $\mathcal{R}_N$, so that the path of the system between two events can be described by the trajectory line $\boldsymbol{Q}(s) = \left(Q^1(s), \dots, Q^N(s)\right)$ embedded into $\mathcal{R}_N$. Furthermore, the equation of motion $\boldsymbol{Q}(t) = \left(Q^1(t), \dots, Q^N(t)\right)$ and the equation of distribution $\boldsymbol{Q}(x) = \left(Q^1(\boldsymbol{x}), \dots, Q^N(\boldsymbol{x})\right)$ of the system have both homomorphic relation to the trajectory. Please note that:

- the time is symbolized as $t \equiv x^0$ and the four coordinates of time and geographic space as $\boldsymbol{x} = (x^0, x^1, x^2, x^3)$ and constitute the usual (Pythagorean) base of the 4-dimensional Euclidean space $\mathcal{E}_4$ of the habitat (plus time) of the system,
- the indexes corresponding to $\mathcal{R}_N$ are symbolized by lower-case Latin letters and take values from 1 to N and the indexes to $\mathcal{E}_4$, by lower-case Greek letters having values from 0 to 3.

Each component of the equation of motion is connected by the corresponding component of the equation of distribution via a density relation:

$$Q^i(x^0) = \int\limits_{\Phi(x^0)} Q^i(\boldsymbol{x})d\Phi\,, d\Phi = dx^1dx^2dx^3 \,:\, i = 1,\dots,N \tag{10}$$

where $\Phi(t)$ is the geographic volume (or surface) of the habitat of the system, which can vary with time.

To a system to be deterministic, its Lagrangian (its state) is given by a general function:

$$L = L\left(Q^1(t),\dots,Q^N(t),\frac{dQ^1(t)}{dt},\dots,\frac{dQ^N(t)}{dt}\right) = L\left(\boldsymbol{Q}(t),\frac{d\boldsymbol{Q}(t)}{dt}\right) \tag{11}$$

so as to include not only the event of the system for every instant, but also the tendency for the event that will follow at the next instant. Consequently, its Lagrangian density is given by:

$$\Lambda = \Lambda\left(\boldsymbol{Q}(\boldsymbol{x}),\frac{\partial\boldsymbol{Q}(\boldsymbol{x})}{\partial x^\mu}\right) \tag{12}$$

hence, according to relation (10), the connection between the two expressions of the Lagrangian is given by:

$$L\left(\boldsymbol{Q}(x^0),\frac{d\boldsymbol{Q}(x^0)}{dx^0}\right) = \int\limits_{\Phi(x^0)} \Lambda\left(\boldsymbol{Q}(\boldsymbol{x}),\frac{\partial\boldsymbol{Q}(\boldsymbol{x})}{\partial x^\mu}\right)d\Phi\,, d\Phi = dx^1dx^2dx^3 \tag{13}$$

The necessity of using a second axiom in constructing the model (the first being the determination of the space of reference as a Riemannian space), is the relation of the trajectory to the space itself. For that, all the intermediate states by which the system goes through between some initial and final state, should be considered, that is the total imprint (the existence) of the system between any initial t_A and final t_B moment of time, namely the action of the system. The uniqueness of the action of a system could be guaranteed by considered it to take a stationary value:

$$S = \int\limits_{t_A}^{t_B} L\left(\boldsymbol{Q}(t),\frac{d\boldsymbol{Q}(t)}{dt}\right)dt \Rightarrow \delta S = \delta\int\limits_{t_A}^{t_B} L\left(\boldsymbol{Q}(t),\frac{d\boldsymbol{Q}(t)}{dt}\right)dt = 0 \tag{14}$$

If the Lagrangian is expressed by a positive definite quadratic form, so that $L(t) > 0\ \forall t$, this stationary value is a minimum. In this case, it can be noticed that: $\delta\big(L(t)\big)^2 = 2L(t)\delta L(t)$ and $\delta\delta\big(L(t)\big)^2 = \delta\big(2L(t)\delta L(t)\big) = 2\big(\delta L(t)\big)^2 + 2L(t)\delta\delta L(t)$, hence:

$$\delta L(t) = 0 \Leftrightarrow \delta\big(L(t)\big)^2 = 0\ \wedge\ \delta\delta L(t) > 0 \Leftrightarrow \delta\delta\big(L(t)\big)^2 > 0 \tag{15}$$

By its nature, the Lagrangian must be a geometric object of the space of reference, since the state of the system should be totally described by means of $\mathcal{R}_N$ and in a manner independent to the choice of the observer and finally, it needs to be a differential of the first order. Since the only such object is the elementary length of the trajectory, it can be deduced, by using relations (10) and (15), that:

$$\delta S = \delta m \int_{s_A}^{s_B} ds = \delta m \int_{s_A}^{s_B} \left(g_{ij}(\boldsymbol{Q}) \frac{dQ^i}{ds} \frac{dQ^j}{ds} \right) ds = 0 \tag{16}$$

where m is a real constant. The application of the Euler-Lagrange equations to relation (16) produces the equation of the trajectory which coincides to the geodesics of $\mathcal{R}_N$:

$$\begin{gathered} \frac{d^2Q^i(s)}{(ds)^2} + \Gamma^i_{jk} \frac{dQ^j(s)}{ds} \frac{dQ^k(s)}{ds} = 0\,, i = 1, \dots, N \\ \Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial Q^k} + \frac{\partial g_{mk}}{\partial Q^j} - \frac{\partial g_{jk}}{\partial Q^m} \right) \end{gathered} \tag{17}$$

where the quantities $\Gamma^i_{jk} \equiv \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the components of the Christoffel symbol of the second kind, following the symbolism use in Physics (i.e., Landau and Lifshitz (1980) or Talman (2007)). Simultaneously, relations (14) and (16) produce the expression of Lagrangian which, by expanding it to a Taylor series, becomes:

$$L = m \frac{ds}{dt} \to L = \frac{1}{2} m\, g_{ij}(\boldsymbol{Q}) \frac{dQ^i}{dt} \frac{dQ^j}{dt} \tag{18}$$

The application of the Euler-Lagrange equation to relation (18) produces the constitutional equation of motion of the system:

$$\frac{d^2Q^i(t)}{(dt)^2} + \Gamma^i_{jk} \frac{dQ^j(t)}{dt} \frac{dQ^k(t)}{dt} = 0\,, i = 1, \dots, N \tag{19}$$

and furthermore, establishes the homomorphic (linear) transformation between s and t.

The action of the system can be also expressed by its Lagrangian density in relation (13):

$$S = \int_{t_A}^{t_B} \int_{\Phi(t)} \Lambda \left(\boldsymbol{Q}(\boldsymbol{x}), \frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial x^\mu} \right) d\Phi dt \Rightarrow \delta S = \delta \int_{\Omega_A}^{\Omega_B} \Lambda \left(\boldsymbol{Q}(\boldsymbol{x}), \frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial x^\mu} \right) d\Omega = 0 \tag{20}$$

where $d\Omega = dx^0 dx^1 dx^2 dx^3$ is the four-volume of $\mathcal{E}_4$, in its usual base. The derivation of the Lagrangian density is produced in the same manner as the Lagrangian of relation (18), by noticing that all the inner products of the unit vectors of $\mathcal{E}_4$ are $dx^\mu dx^\lambda = \delta^{\mu\lambda}$.

$$\Lambda = \frac{1}{2} m\, g_{ij}(\boldsymbol{Q}) \frac{dQ^i}{d\Omega} \frac{dQ^j}{d\Omega} \Rightarrow \Lambda = \frac{1}{2} m\, g_{ij}(\boldsymbol{Q}) \frac{\partial Q^i}{\partial x^\mu} \frac{\partial Q^j}{\partial x^\mu} \tag{21}$$

The application of Euler-Lagrange equations to (21), leads to the constitutional equation of distribution of the system:

$$\frac{\partial^2 Q^i(\boldsymbol{x})}{\partial x^\mu \partial x^\mu} + \Gamma^i_{jk} \frac{\partial Q^j(\boldsymbol{x})}{\partial x^\mu} \frac{\partial Q^k(\boldsymbol{x})}{\partial x^\mu} = 0\,, i = 1, \dots, N \tag{22}$$

Until now, the only unknown factor of equations (19) and (22) are the components of the metric tensor of $\mathcal{R}_N$, which can be determined by the principle of Equivalence (the third axiom), which states that the external influence (force) acting on a system is equivalent to a corresponding curvature of its reference, in such a manner that an inertial system (with no force acting on it) has zero curvature and its space of reference is reduced to a N-dimensional Euclidean space. The curvature of space is expressed by the Riemann-Christoffel tensor:

$$R^i_{jkm} = \frac{\partial \Gamma^i_{jm}}{\partial Q^k} - \frac{\partial \Gamma^i_{jk}}{\partial Q^m} + \Gamma^i_{nk}\Gamma^n_{jm} - \Gamma^i_{nm}\Gamma^n_{jk} \tag{23}$$

so that in the case of a Euclidian space, there is a base, namely the usual base $\boldsymbol{U} = \{U^1, \dots, U^N\}$, where the components of the metric tensor take the form: $g_{ij}(\boldsymbol{U}) = \delta_{ij}$ and, when referred in the usual base, $\Gamma^i_{jk} = 0$ and $R^i_{jkm} = 0$. Therefore, for an inertial system, when referred in the usual base, the components of the equation of motion reduce to the following linear equations:

$$\frac{d^2U^i(t)}{(dt)^2} = 0 \;\Rightarrow U^i(t) = A^i t + B^i, i = 1, \dots, N \tag{24}$$

where A^i and B^i are real constants. Moreover, the components of the equation of distribution are given as solutions of the four-dimensional Laplace equation:

$$\sum_{\mu=0}^{3} \frac{\partial^2 U^i(\boldsymbol{x})}{\partial x^\mu \partial x^\mu} = 0\,, i = 1, \dots, N \tag{25}$$

Evidently, all the components of the equation of motion of an inertial system, for any acceptable base, are uniformly monotonic functions of time.

3. The special case of one-dimensional inertial population system

An N-dimensional population system in general, can be defined by the Kolmogorov equations:

$$\frac{dQ^i(t)}{dt} = f^i\big(Q^1(t), \dots, Q^N(t)\big), i = 1, \dots, N \tag{26}$$

that is, there is some base of its space of reference, namely the natural base $\boldsymbol{Q} = \{Q^1, \dots, Q^N\}$, so that when the components of the equation of motion are described in this base, would be solutions of relation (26). In the case of a one-dimensional inertial system, one can assume, for simplicity, that the equation of motion is reduced to the Malthus equation:

$$\frac{dQ(t)}{dt} = AQ(t) \tag{27}$$

where A is a real constant. Since the system is inertial, there is another base, namely the usual base, by which the equation of motion takes the linear form: $U(t) = At + B$, hence the usual and the natural base of the system are connected by the following acceptable transformation:

$$Q(U) = exp(U) \Leftrightarrow U(Q) = ln(Q)\,, \forall Q > 0 \tag{28}$$

Therefore, the complexity of the differential equations leading to the equation of distribution, as described by the natural base, is reduced considerably:

$$\sum_{\mu=0}^{3}\frac{\partial^2 U(\boldsymbol{x})}{\partial x^{\mu}\partial x^{\mu}}=0\Rightarrow Q(\boldsymbol{x})=exp\big(U(\boldsymbol{x})\big) \tag{29}$$

In the remainder of this paragraph some partial solutions of relation (29) are presented which are compared to field statistical data collected for varies cities around the World. The graphic representations of these data were taken from the paper by Bertaud and Malpezzi (2003), supported by the World Bank. This data connects the population density for each city with a polar radius, taken at some fixed instant of time (the exponential regression lines incorporated in the graphs of the cities are calculated by the initial authors).

By assuming that the distribution of the population density, remains unchanged for all azimuthal angles, an isotropic partial solution of relation (29) is derived. In this case, the equation of distribution is given by a function of the form: $Q = Q(t,\rho)$ and relation (29) becomes:

$$\frac{\partial^2 U(t,\rho)}{\partial t\partial t}+\frac{\partial^2 U(t,\rho)}{\partial \rho\partial \rho}+\frac{1}{\rho}\frac{\partial U(t,\rho)}{\partial \rho}=0 \tag{30}$$

A solution of relation (30) is derived by considering $U(t,\rho)=f(t)+g(\rho)$, so that:

$$\frac{d^2 f(t)}{(dt)^2}-K=0\Rightarrow f(t)=\frac{K}{2}(t)^2+C_1 t+C_2$$

$$\frac{d^2 g(\rho)}{(d\rho)^2}+\frac{1}{\rho}\frac{dg(\rho)}{\partial \rho}+K=0\Rightarrow g(\rho)=-\frac{K}{4}(\rho)^2+C_3 ln(\rho)+C_4$$

where K and C_m are random real constants and the equation of distribution for the natural base becomes:

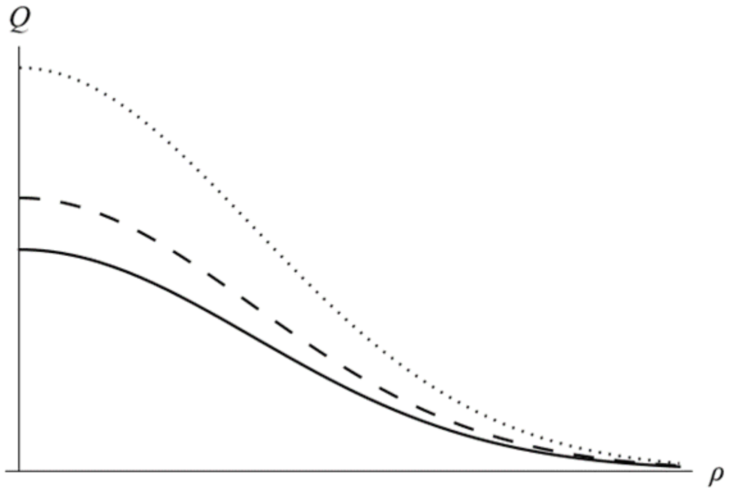
$$Q(t,\rho)=exp(C_2+C_4)(\rho)^{C_3}exp\left(\frac{K}{2}(t)^2+C_1 t-\frac{K}{4}(\rho)^2\right)$$

The first observation for the above relation is that at the point $\rho = 0$, the population density at this point becomes zero or infinity for all instants of time, therefore the corresponding constant should take the value $C_3 = 0 \Rightarrow (\rho)^0 = 1$. Moreover: $Q(0,0) = exp(C_2 + C_4)$, so that the final form of the equation of distribution becomes:

$$Q(t,\rho)=Q(0,0)exp\left(\frac{K}{2}(t)^2+Ct-\frac{K}{4}(\rho)^2\right) \tag{31}$$

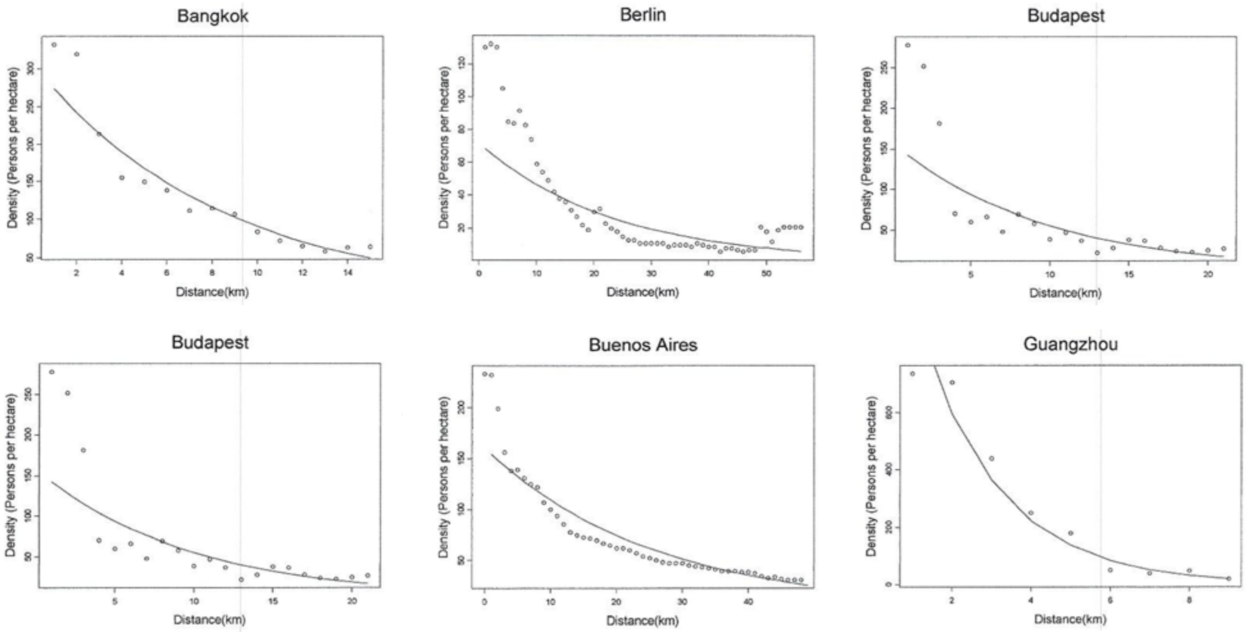
where K and C are real constants estimated empirically, simulating the behavior of relation (1) and (3). Indeed, it can be noticed that for some fixed instant of time, relation (31) takes the form: $Q(t = constant,\rho) = Q(\rho = 0)exp(C_1(\rho)^2 + C_2)$ which, with the proper choice of the real constants C and K, its behavior approximates relation (1), for middle range values of the radius and, also, coincides to the general relation (3) by giving the index m the values $m = 0, 2$.

Figure 2. Representation of relation (31) for $Q(0,0) > 0$, $K > 0$ and $C \geq 0$. Three values of time $t_1 > t_2 > t_3$ are depicted, with solid, dashed and dotted lines respectively, where ρ is the polar radius and $Q = Q(t, \rho)$ is the population density (right: 3D distribution for some fixed time).



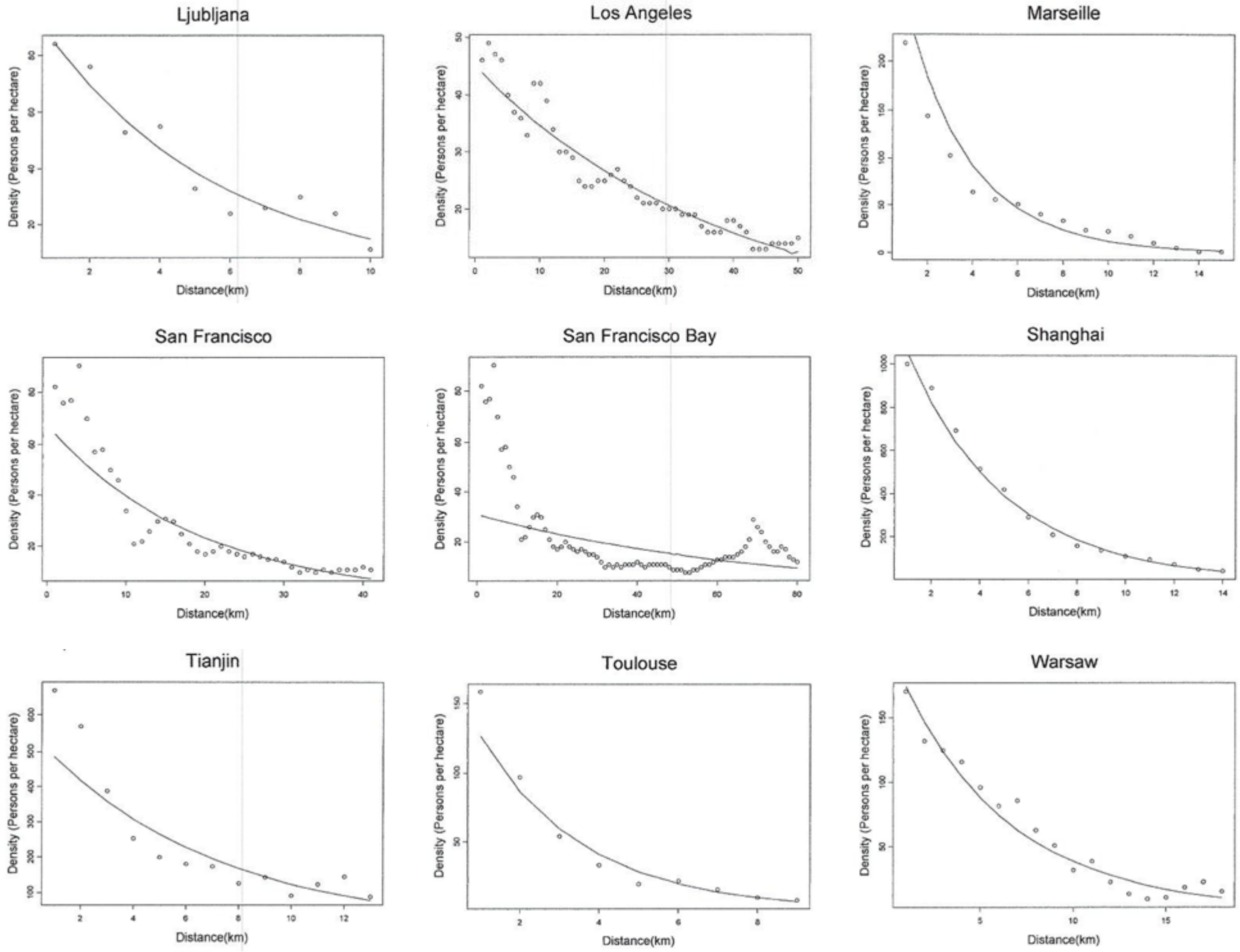
Source: Author's representation

Figure 3. Field data of various cities connecting polar radius to population density, approximating a solution where relation (31) and Figure 2 is predominant.



Source: Bertaud and Malpezzi (2003)

Figure 3 (continued). Field data of various cities connecting polar radius to population density, approximating a solution where relation (31) and Figure 2 is predominant.



Source: Bertaud and Malpezzi (2003)

Another solution of relation (30) is derived by considering $U(t,\rho) = f(t)g(\rho)$, so that:

$$\frac{d^2 f(t)}{(dt)^2} - Kf(t) = 0 \Rightarrow f(t) = C_1 exp\left(t\sqrt{K}\right) + C_2 exp\left(-t\sqrt{K}\right)$$

$$\frac{d^2 g(\rho)}{(d\rho)^2} + \frac{1}{\rho}\frac{dg(\rho)}{\partial \rho} + Kg(\rho) = 0 \Rightarrow g(\rho) = C_3 J_0\left(\rho\sqrt{K}\right) + C_4 Y_0\left(\rho\sqrt{K}\right)$$

where K and C_m are real constants estimated empirically and the equation of distribution for the natural base becomes:

$$Q(t,\rho) = exp\Big(C_1 exp\left(t\sqrt{K}\right) J_0\left(\rho\sqrt{K}\right) + C_2 exp\left(-t\sqrt{K}\right) J_0\left(\rho\sqrt{K}\right) + \\ + C_3 exp\left(t\sqrt{K}\right) Y_0\left(\rho\sqrt{K}\right) + C_4 exp\left(-t\sqrt{K}\right) Y_0\left(\rho\sqrt{K}\right)\Big) \tag{32}$$

The choices of the constants lead to two distinct behaviors for the population distribution:

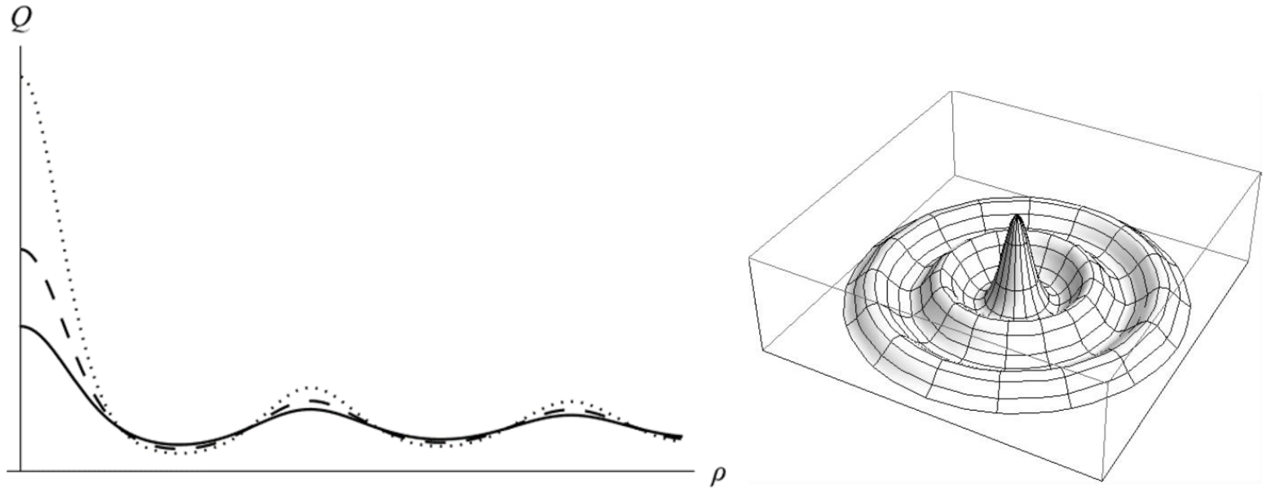
$$Q(t,\rho) = exp\left(C_1 exp\left(t\sqrt{K}\right) J_0\left(\rho\sqrt{K}\right)\right) \tag{33}$$

where $J_0\left(\rho\sqrt{K}\right)$ is the Bessel function of the first kind with integer number 0, and

$$Q(t,\rho) = exp\left(C_1 exp\left(t\sqrt{K}\right)Y_0\left(\rho\sqrt{K}\right)\right) \tag{34}$$

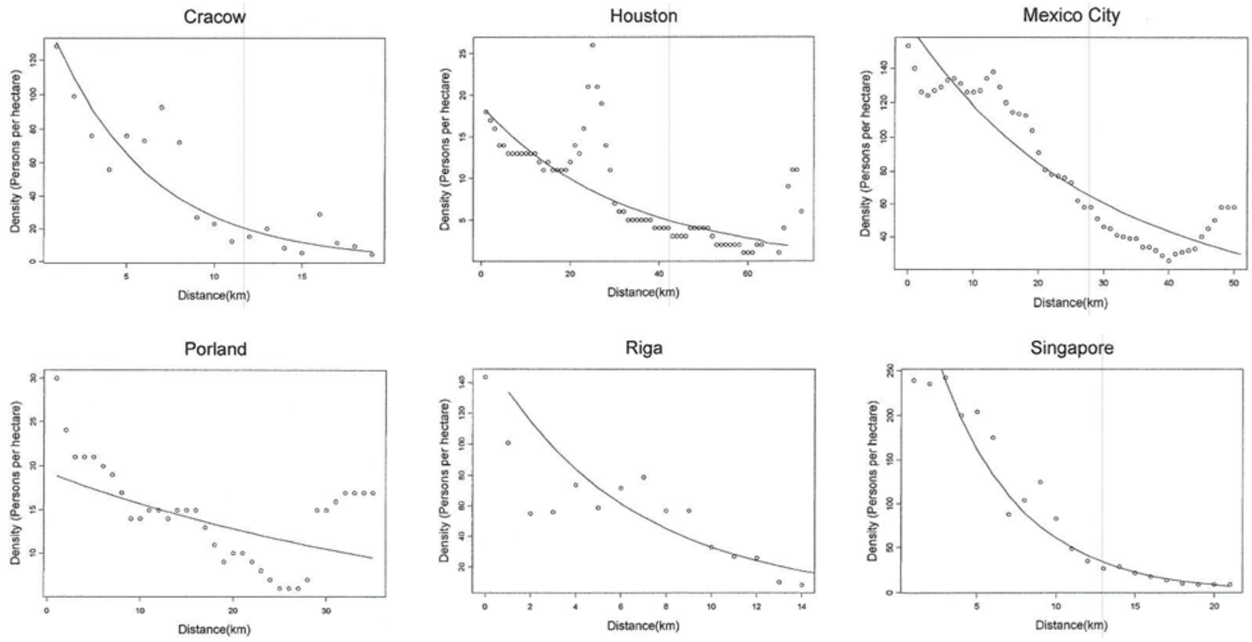
where $Y_0\left(\rho\sqrt{K}\right)$ is the Bessel function of the second kind with integer number 0.

Figure 4. Representation of relation (33) for $C_1 > 0$ and $K > 0$. Three values of time $t_1 > t_2 > t_3$ are depicted, with solid, dashed and dotted lines respectively, where ρ is the polar radius and $Q = Q(t,\rho)$ is the population density (right: 3D distribution for some fixed time).



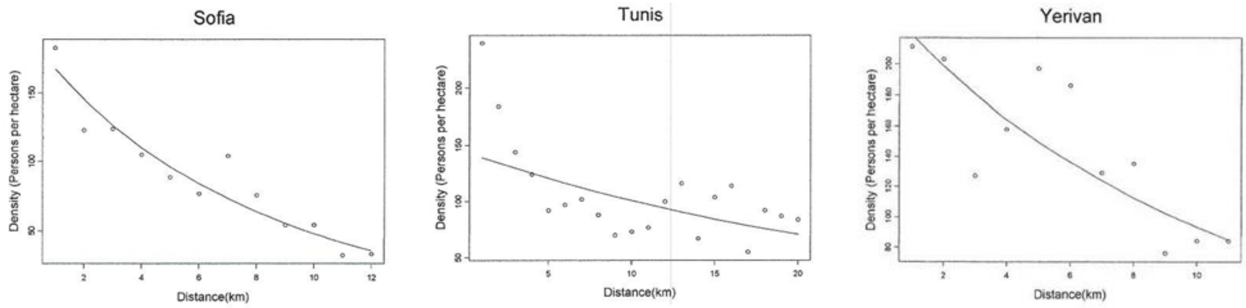
Source: Author's representation

Figure 5. Field data of various cities connecting polar radius to population density, approximating a solution where relation (33) and Figure 4 is predominant.



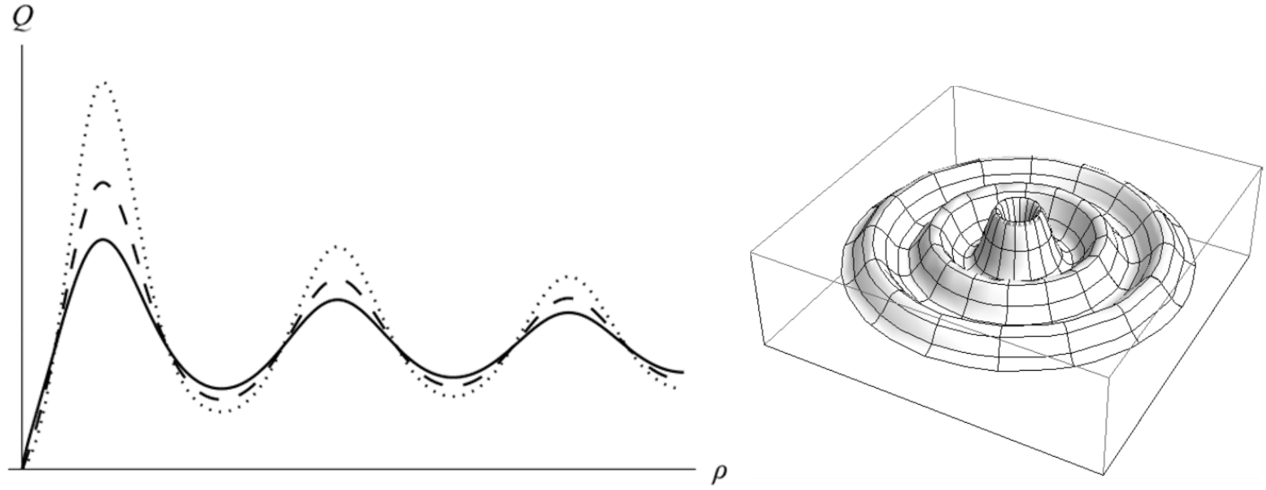
Source: Bertaud and Malpezzi (2003)

Figure 5 (continued). Field data of various cities connecting polar radius to population density, approximating a solution where relation (33) and Figure 4 is predominant.



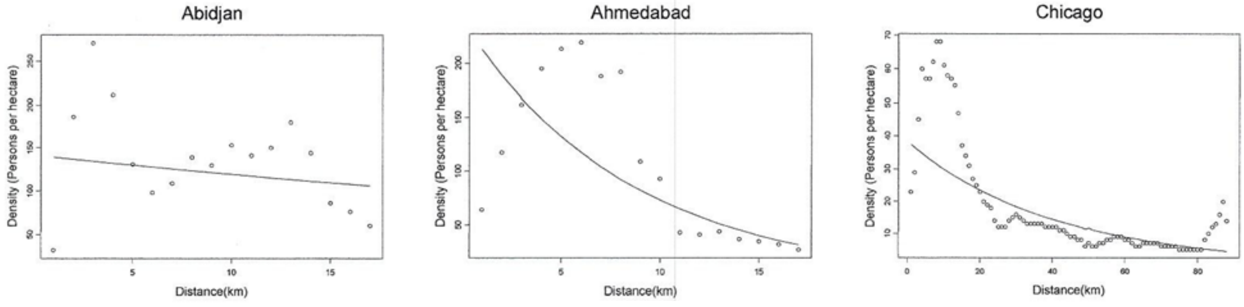
Source: Bertaud and Malpezzi (2003)

Figure 6. Representation of relation (34) for $C_1 > 0$ and $K > 0$. Three values of time $t_1 > t_2 > t_3$ are depicted, with solid, dashed and dotted lines respectively, where ρ is the polar radius and $Q = Q(t, \rho)$ is the population density (right: 3D distribution for some fixed time).



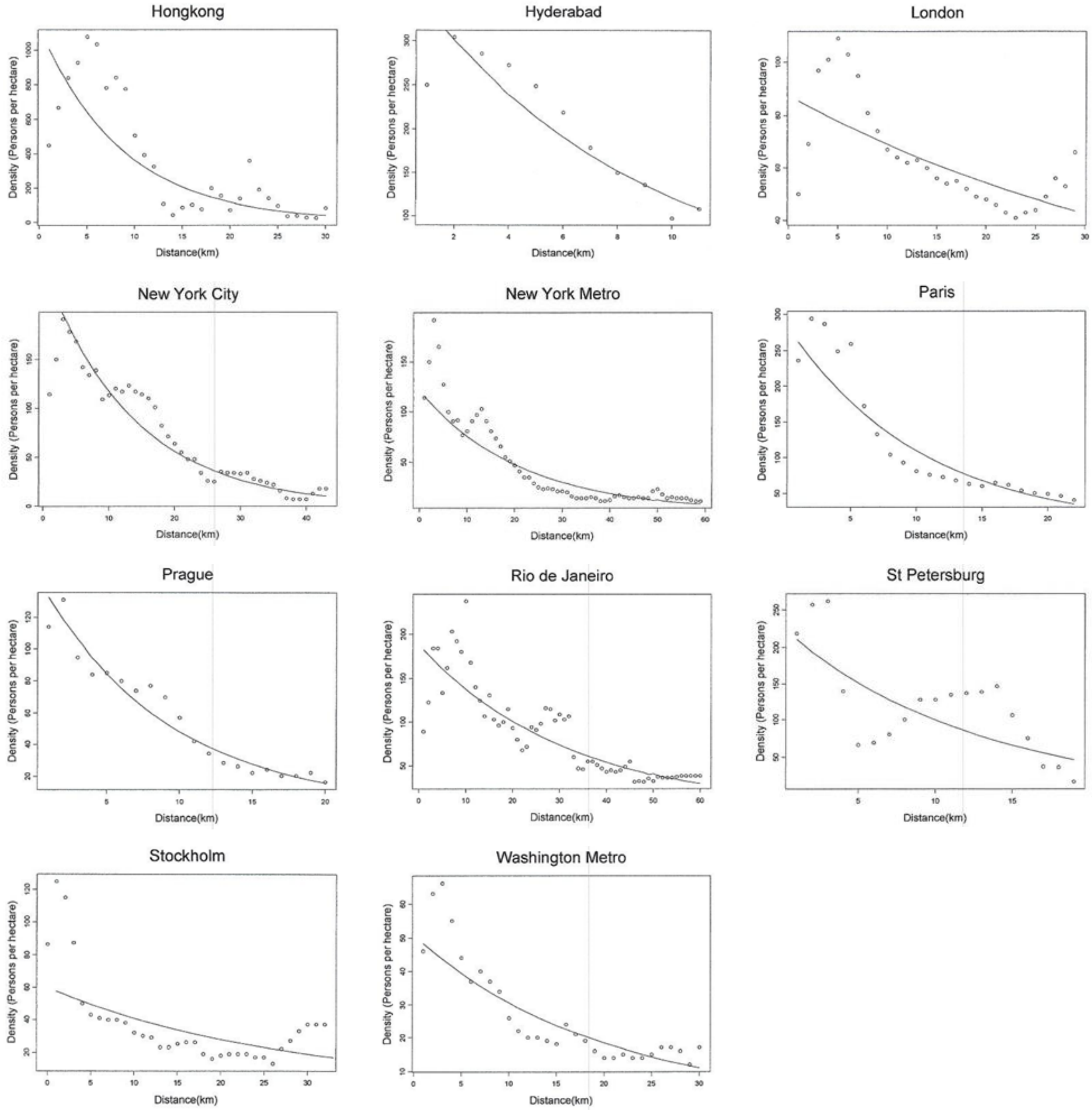
Source: Author's representation

Figure 7. Field data of various cities connecting polar radius to population density, approximating a solution where relation (34) and Figure 6 is predominant.



Source: Bertaud and Malpezzi (2003)

Figure 7 (continued). Field data of various cities connecting polar radius to population density, approximating a solution where relation (34) and Figure 6 is predominant.



Source: Bertaud and Malpezzi (2003)

It can be noticed that relation (32), as well as relations (33) and (34), when restricted to a constant instant of time: $Q = Q(t = constant, \rho)$, appear to have similar behavior to the general equation of relation (3), for some small intervals of the polar radius. Indeed, the polynomial function of the exponent of relation (3) approximate the wave-like behavior of the Bessel functions for a small interval of the polar radius. This approximation is analogous to the Taylor series expansion of a trigonometric function.

A set of partial solutions of relation (29) is produced by emphasizing the homogeneous character of population in a two-dimensional geographic space, so that relation (29) takes the form:

$$\frac{\partial^2 U(t,x,y)}{\partial t \partial t} + \frac{\partial^2 U(t,x,y)}{\partial x \partial x} + \frac{\partial^2 U(t,x,y)}{\partial y \partial y} = 0 \tag{35}$$

A solution of relation (35) can be of the form: $U(t,x,y) = f(t) + g(x) + h(y)$:

$$\frac{d^2 f(t)}{(dt)^2} - K_1 = 0 \Rightarrow f_1(t) = \frac{K_1}{2}(t)^2 + C_1 t + C_2$$

$$\frac{d^2 g(x)}{(dx)^2} - K_2 = 0 \Rightarrow g(x) = \frac{K_2}{2}(x)^2 + C_3 x + C_4$$

$$\frac{d^2 h(y)}{(dy)^2} - K_3 = 0 \Rightarrow h(y) = \frac{K_3}{2}(y)^2 + C_5 y + C_6$$

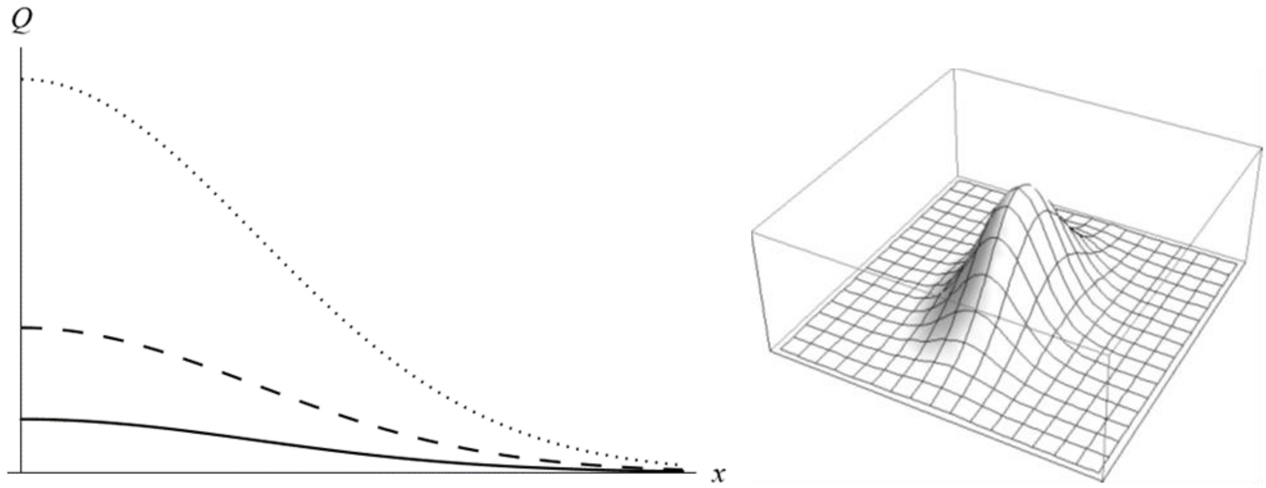
$$K_1 + K_2 + K_3 = 0$$

and $Q(t,x,y) = exp\big(U(t,x,y)\big)$:

$$Q(t,x,y) = Q(0,0,0)exp\left(\frac{K_1}{2}(t)^2 + C_1 t + \frac{K_2}{2}(x)^2 + C_3 x + \frac{K_3}{2}(y)^2 + C_5 y\right) \tag{36}$$

where K_m and C_n are real constants estimated empirically, which has a distribution behavior similar to that of relation (31), but not necessarily isotropic.

Figure 8. Representation of relation (36) for $K_1 > 0$, $K_2 < 0$ and $K_3 < 0$. Three values of time $t_1 > t_2 > t_3$ are depicted, with solid, dashed and dotted lines respectively, where x is one axis and $Q = Q(t, x, y = constant)$ is the population density (right: 3D distribution for some fixed time).



Source: Author's representation

Finally, a solution of (35) of the form: $U(t,x,y) = f(t)g(x)h(y)$ is given:

$$\frac{d^2 f(t)}{(dt)^2} - K_1 f(t) = 0 \Rightarrow f(t) = C_1 exp\big(t\sqrt{K_1}\big) + C_2 exp\big(-t\sqrt{K_1}\big)$$

$$\frac{d^2g(x)}{(dx)^2} - K_2g(x) = 0 \Rightarrow g(x) = C_3exp\left(x\sqrt{K_2}\right) + C_4exp\left(-x\sqrt{K_2}\right)$$

$$\frac{d^2h(y)}{(dy)^2} - K_3h(y) = 0 \Rightarrow h(y) = C_5exp\left(y\sqrt{K_3}\right) + C_6exp\left(-y\sqrt{K_3}\right)$$

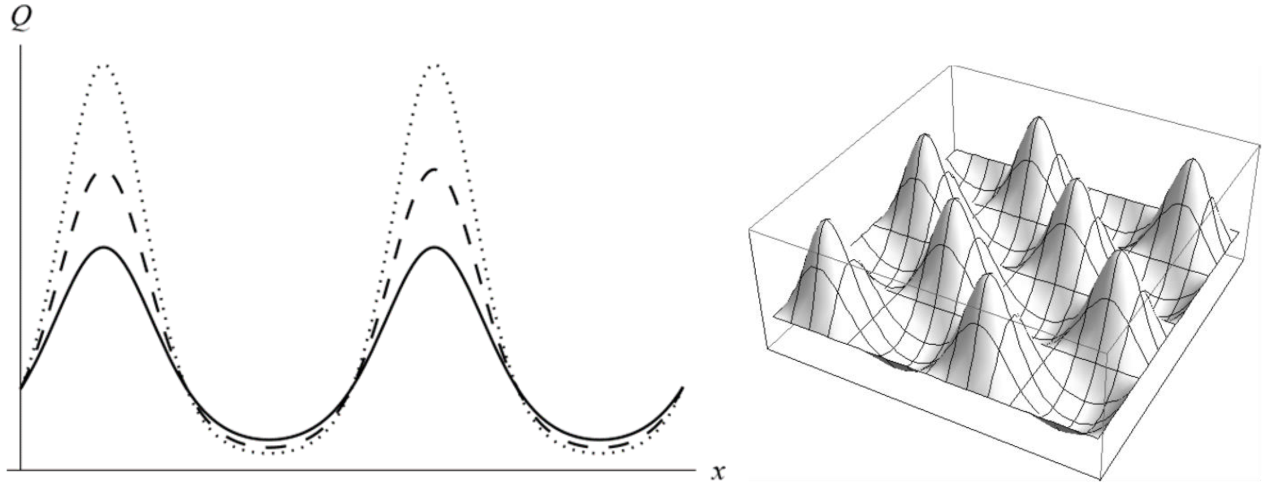
$$K_1 + K_2 + K_3 = 0$$

For which the equation $Q(t,x,y) = exp\big(U(t,x,y)\big)$ produces some interesting results when the real constants C, C_i and K_i, estimated empirically, take the values $C > 0$, $K_1 > 0$, $K_2 < 0$ and $K_3 < 0$:

$$Q(t,x,y) = exp\left(Cexp\left(t\sqrt{K_1}\right)sin\left(x\sqrt{|K_2|}\right)sin\left(y\sqrt{|K_3|}\right)\right) \tag{37}$$

which demonstrates the existence of city-villages, in relevance to the works by Griffith and Wong (2007).

Figure 9. Representation of relation (37) for three values of time $t_1 > t_2 > t_3$ with solid, dashed and dotted lines respectively, where x is one axis and $Q = Q(t,x,y = constant)$ is the population density (right: 3D distribution for some fixed time).



Source: Author's representation

It should be emphasized that the equations describing the population distribution in this paragraph are only some simple partial solutions of the Laplace equation which is a linear PDE and hence the principle of superposition is applied. Therefore, in general, all linear combinations of relations (31), (33) and (34) satisfy relation (30) and every such combination can describe the isotropic behavior of the population. Similarly, the "homogeneous" behavior of a population is given by any linear combinations of relation (36) and (37) since all constitute solutions of relation (35).

Evidently, a city rarely can be considered as an inertial system (which is the subject of this paragraph), but there are some reasons in justifying the categorization proposed in Figures 3, 5 and 7. The more apparent and simpler reason is that the inner (intrinsic) property of the exponential growth of the population seems prevalent, in some cases, compared to the external influences imposed upon the city. A second reason is that the field data of the above Figures were "snapshots" of the population distribution and not a spatiotemporal record of its holistic behavior. As it can be seen in the next paragraph, in many cases (Figure 12) the external influence is revealed when the "snapshot" is taken at the proper instant, or better yet, when multiples "snapshots" are taken of the same city at different instants and compared to each other, as in Bergmann (2019).

In Figures 3, 5 and 7 the following remarks can be made:

- This Figures along with Figure 12 in the next paragraph, present the graphs of observational data, for some instance of time, of the population density of cities for specific distances from the center of each city (along the polar radius from the city center outward) as points of the distance-density coordinate system. They also include an exponential regression of these data (solid line), corresponding to the Clark equation of relation (1).
- The observational data of the cities presented in Figure 3, generally indicate a) a slow initial (small values of radius) descent of the population density, b) a rapid descent of the density for middle range values of the radius and c) a slow descent for large values of the radius, which characterize the bell-shape function of relation (31).
- Evidently, the behavior of the observation data of the cities presented in Figures 5 and 7 are qualitatively radically different of the ones in Figure 3, in the sense that the population density follows a wave-like pattern (repetitive increase and decrease motif) as the polar radius increases, with the local maxima diminishing with each cycle. The absolute or a local minimum, occurs at the center of the cities ($\rho = 0$), presented in Figure 5, as described by relation (34). One can observe that this central open space (low population density) occurred in the majority of Ancient Greece cities (agora - center marketplace).
- On the contrary, in the cases of Figure 7, the absolute maximum of each city occurs at the city center, in addition to the above-mentioned wave-like pattern, as described in relation (33).

The theoretical solutions of temporal behavior of the population systems examined in this paragraph, depicted in Figures 2, 4, 6, 8 and 9, indicate that as a system develops, its population density increases in areas of local maxima, and decreases (in a smaller ratio) in areas of local minima. This behavior can be noticed in the observative data presented in Bergmann (2019).

4. The special case of one-dimensional dynamic population system of constant curvature

The estimation of the macroscopic behavior of a system inevitably includes some generalizations and approximations concerning the nature of the totality of the external influences acting on this system, which mainly involves all other systems that interact or compete with the first. One such approximation which, in addition, can easily be mathematically formulated, is the hypothesis of the homogeneous and isotropic external influence (force). By this hypothesis, it is assumed that the influence of a system toward its environment (and the reaction of the environment toward the system) is proportional to the magnitude of this system at any given time. Therefore, any variation of the components of the base of the system (one or many populations, primary production, GNP etc.) would be met by a corresponding variation of the reaction (force) of the environment both quantitatively (homogeneous) and qualitatively (isotropic), following, in the long term, the vector of the variation in magnitude and direction respectively. By the principle of Equivalence, the character of the external influence corresponds to a similar character of the space of reference of the system, hence the latter should be homogeneous and isotropic, that is a space of constant curvature. The first case, that of zero curvature (zero external influence) was examined in paragraph 3 and in this paragraph the positive and negative constant curvature will be investigated.

The positive constant curvature λ of a space is defined as:

$$\lambda = \frac{1}{(R)^2} \tag{38}$$

where R is the radius of curvature of this space and it is a positive real constant. The metric of a one-dimensional system, where its space of reference has a usual base of the form: $\boldsymbol{U} = \{U\}$ can be calculated by using an additional component, say U^1 and start with Pythagorean metric in two dimensions: $(ds)^2 = (dU)^2 + (dU^1)^2$, with the additional constrain $(R)^2 = (U)^2 + (U^1)^2$. So, the metric takes the form:

$$(dU^1)^2 = \frac{(U)^2}{(R)^2 - (U)^2}(dU)^2 \Rightarrow (ds)^2 = (dU)^2 + \frac{(U)^2}{(R)^2 - (U)^2}(dU)^2$$

and finally:

$$(ds)^2 = \frac{(R)^2}{(R)^2 - (U)^2}(dU)^2 \tag{39}$$

Since the metric tensor is known, the equations of motion and distribution can be derived directly, but a transformation can be applied which simplifies the calculations. An intermediate base $\boldsymbol{W} = \{W\}$ can be found so that, the metric tensor takes the unit value when described by this new base:

$$(ds)^2 = \frac{(R)^2}{(R)^2 - (U)^2}(dU)^2 = (dW)^2 \Rightarrow W(U) = Ratan\left(\frac{U}{\sqrt{(R)^2 - (U)^2}}\right)$$

and the requested transformation is:

$$U(W) = Rsin\left(\frac{W}{R}\right) \tag{40}$$

It should be noticed that the metric (39) is not Pythagorean since the transformation between the bases U and W is acceptable for all the domain of $U(W)$ except for the values $U = \pm R$. This exception is indicative of a non-Euclidian space of reference and, hence, of the existence of external force.

Since the metric tensor for the base W has a constant, all the Christoffel symbols of the second kind vanish and the equation of motion takes the form:

$$\frac{d^2W(t)}{(dt)^2} = 0 \Rightarrow W(t) = At + B$$

and by applying the combined transformations of relations (28) and (40)

$$\frac{d^2W(t)}{(dt)^2} = 0 \Rightarrow W(t) = At + B \Rightarrow$$

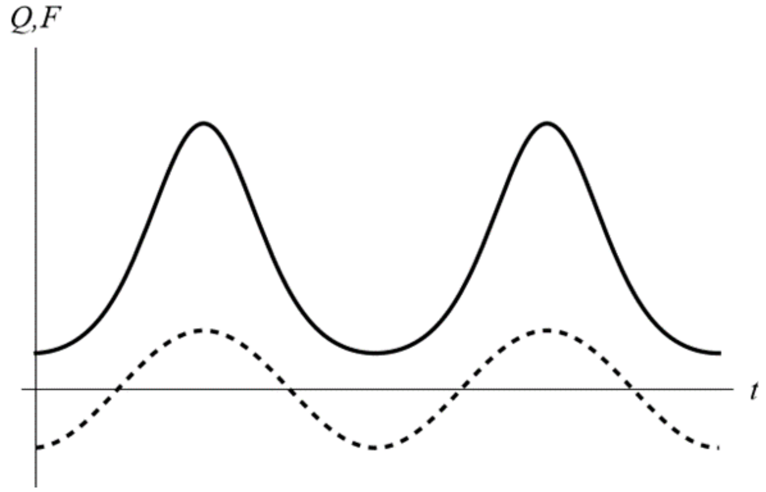
$$\Rightarrow U(t) = Rsin\left(\frac{At + B}{R}\right) \Rightarrow$$

$$\Rightarrow Q(t) = exp\left(Rsin\left(\frac{At + B}{R}\right)\right)$$

where $R > 0$, A and B are real constants. The last equation of motion $Q(t)$ is periodic, with period T and takes its extreme values at $Q_{min}(t = 0) = exp(-R)$ and $Q_{max}(t = T/2) = exp(R)$, hence:

$$Q(t) = exp\left(ln(Q_{max})sin\left(\frac{2\pi}{T}t - \frac{\pi}{2}\right)\right) \tag{41}$$

Figure 10. Representation of the equation of motion (thick line) of relation (41) and the external force (dashed line) of relation (42), where t is time and $Q = Q(t)$ and $F = F(t)$.



Source: Author's representation

Also, the external force acting on the system in the usual base is:

$$F(t) = -\frac{d^2U(t)}{(dt)^2} = ln(Q_{max})sin\left(\frac{2\pi}{T}t - \frac{\pi}{2}\right) \tag{42}$$

The use of this series of transformations can be appreciated when the solutions of the equation of distribution are calculated. Indeed, since for the base W the Christoffel symbols vanish, this solution reduces to:

$$\Delta W(\boldsymbol{x}) = \sum_{\mu=0}^{3} \left(\frac{\partial^2 W(\boldsymbol{x})}{\partial x^\mu \partial x^\mu} \right) = 0 \Rightarrow Q(\boldsymbol{x}) = exp\left(Rsin\left(\frac{W(\boldsymbol{x})}{R} \right) \right) \tag{43}$$

where some solutions of the four-dimensional Laplace equation have already been produced in paragraph 3. As an example, the case of relation (33):

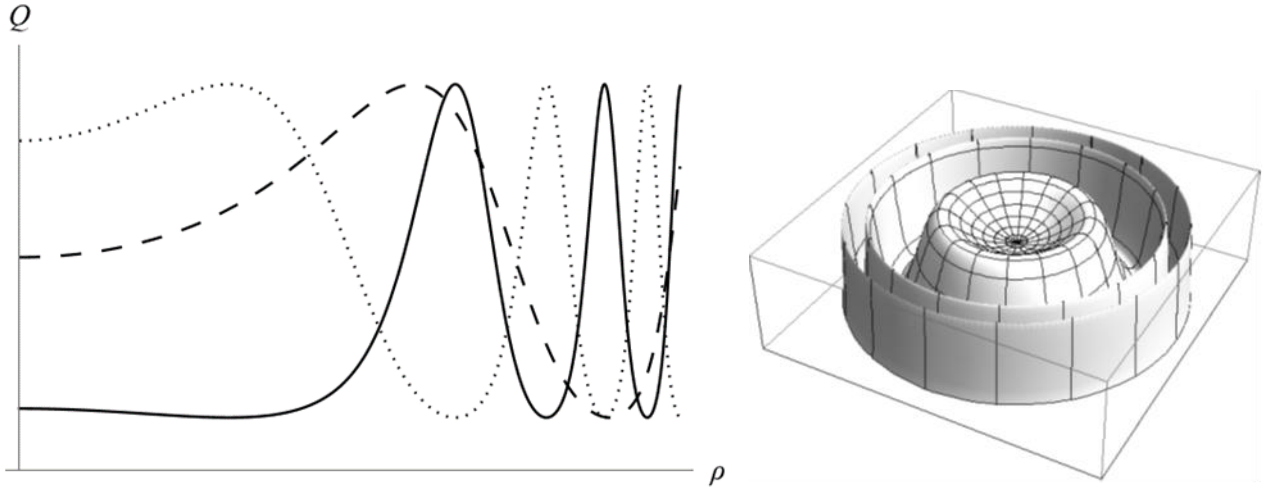
$$\Delta W(t,\rho) = 0 \Rightarrow W(t,\rho) = \frac{1}{R} C_1 exp\big(t\sqrt{K}\big) J_0\big(\rho\sqrt{K}\big)$$

can be considered, producing some interesting results when $K < 0$:

$$Q(\boldsymbol{x}) = exp\left(Rsin\left(\frac{C}{R} cos\left(t\sqrt{|K|}\right) J_0\left(i\rho\sqrt{|K|}\right) \right) \right) \tag{44}$$

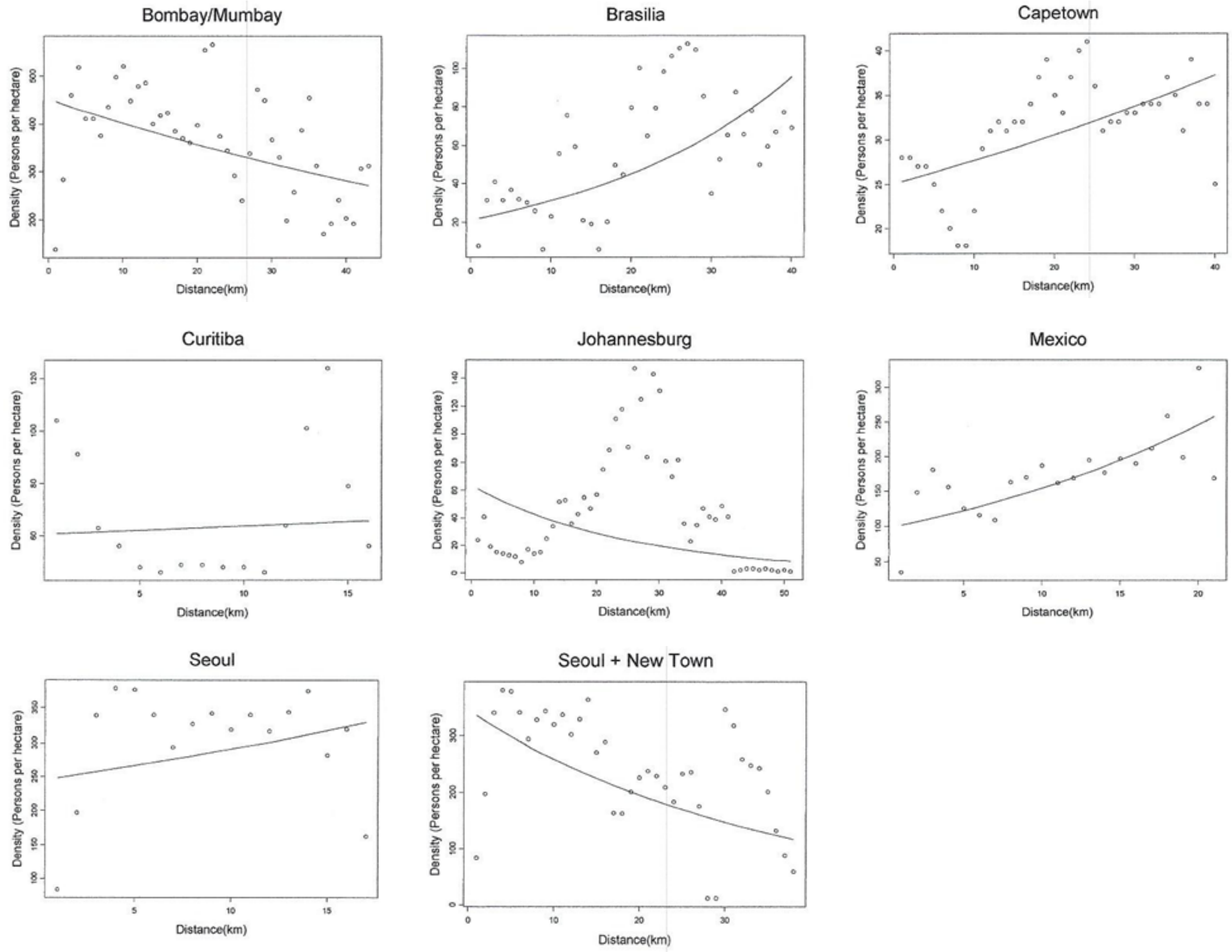
where $K < 0$ and $C < 0$ are real constants estimated empirically and $J_0\left(i\rho\sqrt{|K|}\right)$ is the Bessel function of the first kind with integer number 0.

Figure 11. Representation of relation (44) for three values of time $t_1 > t_2 > t_3$ with solid, dashed and dotted lines respectively where ρ is the polar radius and $Q = Q(t,\rho)$ is the population density (right: 3D distribution for some fixed time). When time reaches the value $t = T/2$, the population distribution follows the reverse path (that is $t_3 \rightarrow t_2 \rightarrow t_1$).



Source: Author’s representation

Figure 12. Field data of various cities connecting polar radius to population density, approximating relation (44) and Figure 12.



Source: Bertaud and Malpezzi (2003)

The negative constant curvature λ of a space can be developed in a similar manner:

$$\lambda = \frac{1}{(iR)^2} \Rightarrow \lambda = -\frac{1}{(R)^2} \tag{45}$$

where R is the radius of curvature of this space and it is a positive real constant. The metric of a one-dimensional system, where its space of reference has a usual base of the form: $\boldsymbol{U} = \{U\}$ can be calculated by using an additional component, say U^1 and start with the Pythagorean metric in two dimensions: $(ds)^2 = (dU)^2 + (dU^1)^2$, with the additional constraint $-(R)^2 = (U)^2 + (U^1)^2$. So, the metric takes the form:

$$(dU^1)^2 = -\frac{(U)^2}{(R)^2+(U)^2}(dU)^2 \Rightarrow (ds)^2 = (dU)^2 - \frac{(U)^2}{(R)^2+(U)^2}(dU^1)^2$$

and finally:

$$(ds)^2 = \frac{(R)^2}{(R)^2+(U)^2}(dU)^2 \tag{46}$$

As in the previous case, an intermediate base $\boldsymbol{W} = \{W\}$ will be used to reduce the metric tensor to take the unit value:

$$(ds)^2 = \frac{(R)^2}{(R)^2 + (U)^2}(dU)^2 = (dW)^2 \Rightarrow W(U) = Rasinh\left(\frac{U}{R}\right)$$

and the requested transformation is:

$$U(W) = Rsinh\left(\frac{W}{R}\right) \tag{47}$$

Since the metric tensor for the base W has a constant, all the Christoffel symbols of the second kind vanish and the equation of motion takes the form:

$$\frac{d^2W(t)}{(dt)^2} = 0 \Rightarrow W(t) = At + B$$

and by applying the transformations of relations (28) and (47)

$$\frac{d^2W(t)}{(dt)^2} = 0 \Rightarrow W(t) = At + B \Rightarrow$$

$$\Rightarrow U(t) = Rsinh\left(\frac{At + B}{R}\right)$$

The equation of motion in the natural base is:

$$Q(t) = exp\left(Rsinh\left(\frac{At + B}{R}\right)\right) \tag{48}$$

where $R > 0$, A and B are real constants and the external force acting on the system in the usual base is given by:

$$F(t) = -\frac{d^2U(t)}{(dt)^2} = -\frac{(A)^2}{R}sinh(At + B) \tag{49}$$

As in the case of positive curvature, the equation of distribution is simplified as:

$$\Delta W(\boldsymbol{x}) = \sum_{\mu=0}^{3}\left(\frac{\partial^2 W(\boldsymbol{x})}{\partial x^\mu \partial x^\mu}\right) = 0 \Rightarrow Q(\boldsymbol{x}) = exp\left(Rsinh\left(\frac{W(\boldsymbol{x})}{R}\right)\right) \tag{50}$$

Although over-simplified, both as a system and as an external influence acting upon it, the simulation of this paragraph can nevertheless deduce some indications about the long-term behavior of population density, as shown in relation (44) and Figure 11, which do not contradict the observational data presented in Figure 12:

- Let us consider a city that, when in an inertial state, behaves as predicted by relation (33). The application of a homogeneous and isotropic external force acts as a “compression” upon the population density, in a manner that this “compression” is analogous to the size of the population (and the population density) at every instant. In the long term, this city follows a circle of ascent, climax and descent, as in Figure 10.

- The wave-like pattern is retained but, in the dynamic state, the wavelength (the polar radius intervals between two consequent local maxima) diminishes as the polar radius increases. On the contrary, the wave width (the difference between the values of a local maximum and the consequent local minimum) is not diminished as the polar radius increases (as in the case of an inertial city).
- The most significant dynamic characteristic is that at every point of the polar radius, the population density increases and decreases circularly over time (see Figures 10 and 11), so that large numbers of the population move from the city center to its periphery and back again, resembling stationary waves.
- The population distribution graphs of the cities presented in Figure 12 are "snapshots" of each city's history. If it is assumed that all these cities are influenced by the same kind of external forces (as the behavior of their local maxima indicates), perhaps each graph in Figure 12 represents a different instant of the history of similar cities. By accepting this hypothesis, the population movement between the center of a city and its periphery can be noticed, in Figure 12.

5. Discussion, conclusion, and directions for future research

The issues that the present paper tries to address, were motivated by a preliminary and rather casual discussion at the Technical Chamber of Greece, concerning the (then) new building regulations for the Metropolitan Areas of Athens and Thessaloniki, namely the height of buildings in different areas of each city, the zoning, the placement and percentage of public open spaces, etc. It has been proven, time and again, that every time any regulation (or any legislation for that matter) ignores the inherent natural properties of the system upon which it is going to be applied, the situation of this system becomes worse than before the application of this regulation - like ignoring gravitation when studying a new building. Evidently, the intimate knowledge of the distribution of a population is essential in Urban and Regional Planning but has much broader social and political implications, especially when studied in combination with other variables.

Since 1951, when Clark published the equation described in relation (1), many equations have been produced (i.e., see relations (2) through (7)) to simulate the behavior of the population density of a city. Most of these equations, especially the ones referring not to a specific city of area, but propose a more general rule, are special case of relation (3), that is, an exponential function with a polynomial of the polar radius as exponent. Hence, the above equations can describe the population density of a city for a specific instant in time, only for some interval of the polar radius (as seen in Figures 3, 5, 7 and 12) and for the case of isotropic expansion of the city. The main characteristic of the approach presented in this paper is the deduction of the equations of distribution by an analytical deterministic general model, which permits the study of both the spatial and temporal behavior of population density. The existence of this general model and its capability to produce specialized systems, permits the study of the behavior of one-dimensional (as in the cases of the present paper) or multidimensional systems, inertial (as in Paragraph 3) or dynamic (as in Paragraph 4), having isotropic (as in relation (30)) or non-isotropic (as in relation (35)) expansion. Moreover, the observational data given in Figures 3, 5, 7 and 12, seem to qualitatively validate the resulting theoretically derived equations.

The analytical derivation of the equation of population distribution could present some additional benefit, even at this early stage of the development of the model. Indeed, the existence of a constitutional differential equation for the population distribution, in the forms of relations (30) or (35), makes the investigation of the alterations of population density of a city, due to topographic morphology (coastline, rivers, mountains, including railroads or highways, etc.) possible, by reducing this subject to a boundary problem applied to the constitutional equations. Furthermore, the knowledge of a spatiotemporal equation of distribution, provides the means of determining present AND FUTURE optimal paths of the propagation of interaction within a city, via the application of variational methods, thus benefiting urban and transportation planning.

Future research on the subject of this paper could include two logical procedures leading to the improvement and refinement of the general simulation described in paragraph 2. Firstly, the case of paragraph 3 should be generalized to include multidimensional population systems, in such a way that every population (variable – dimension) would interact with each other. This procedure would also determine the exact form of the external force acting upon a population system, originating from another analogous system. Apart from the observational density data from interacting cities, the Kolmogorov equation of relation (26) should be used as a theoretical guide. Indeed, the metric tensor of relation (9) should produce an equation of motion of the system that could be reduced to some form of the Kolmogorov equation. Moreover, two special cases of Kolmogorov equations should be investigated, namely the linear N-dimensional first order differential system of equations:

$$\frac{dQ^i(t)}{dt} = \sum_{k=1}^{N} \left(A_k^i Q^k(t) \right) \, , i = 1, \dots, N \tag{51}$$

where A_k^i are real constants and the Lotka-Volterra system (see, for example, Dick (2004), Francisco (2009) or Logan and Wolesensky (2009)):

$$\frac{dQ^1(t)}{dt} = AQ^1(t) + BQ^1(t)Q^2(t)$$
$$\frac{dQ^2(t)}{dt} = CQ^2(t) + DQ^2(t)Q^1(t) \tag{52}$$

where A, B, C and D are real constants, and its generalized form:

$$\frac{dQ^i(t)}{dt} = A_i Q^i(t) + \sum_{k,m=1}^{N} \left(A^i_{km} Q^k(t) Q^m(t)\right), i = 1, \dots, N \tag{53}$$

where A_i and A^i_{km} are real constants. The next, more laborious, procedure should be the incorporation to a multidimensional dynamic system some variables of both population and economic nature.

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